

Algebraic Number Theory

Robert F. Coleman

Lecture 13

Let K_{nr} be the maximal unramified extension of K and $L_\pi = K_\pi K_{nr}$. I showed last time that $L =: L_\pi$ is independent of π using some lemmas that I will prove. I will show L is the maximal Abelian extension of K , K_{ab} , and $\text{Gal}(L/K) \cong \hat{K}^*$ using some results at the beginning of Serre's article.

The norm reciprocity map

Let $r_\pi: K^* \rightarrow \text{Gal}((L/K))$ be the map defined by

$$r_\pi(\pi^n a)(x) = \begin{cases} x^{\sigma^n} & x \in K_{nr} \\ [a^{-1}]_f(x) & x \in E_f, \end{cases}$$

where $f \in \mathcal{F}_\pi$. In particular, $r_\pi(\pi)$ fixes E_f .

Theorem. $r_\pi = r_{\pi'}$.

Proof. Suppose $\phi^\sigma = \phi \circ [u]_f$ and $g = \phi^\sigma f \phi^{-1}$. Let $x \in E_f$. Let $s = r_\pi(\pi')$.

Then

$$(\phi(x))^s = \phi^\sigma([u^{-1}]_f(x))$$

There is a homomorphism $\theta: K^* \rightarrow \text{Gal}(K_{ab}/K)$ such that if $F \subseteq K_{ab}$ is a finite extension of K and $\alpha \in N_K^F(F^*)$, $\theta(\alpha)$ fixes F . Also, if π is a parameter, $\theta(\pi)$ restricted to K_{nr} is Frobenius. §2.2

Lemma. Suppose $K_{nr} \subseteq M \subseteq K_{ab}$ and M_π is the fixed field of $\theta(\pi)$. Then, $M = M_\pi \otimes K_{nr}$ $\text{Gal}(M/K) = \text{Gal}(M_\pi/K) \times \hat{\mathbf{Z}}$.

Proof.

Proposition. Suppose $f: K^* \rightarrow \text{Gal}(M/K)$ such that

1) $f(\pi)|_{K_{n^r}}$ is Frobenius and

2) $f(\pi)|_{M_\pi}$ is trivial.

Then $f(\alpha) = \theta(\alpha)|_M$.

Theorem. $r = \theta$.