Real Analysis

Lecture 12

1 Cauchy Sequences

A sequence $(p_n)_{n \in J}$ in X is said to be **Cauchy** if for every $r > 0 \in \mathbf{R}$, there exists an $N \in J$ such that $|p_n - p_m| < r$ for all $n, m > N$.

Theorem 1.1 *Every Cauchy sequence in a compact metric space converges to a point of X.*

Proof. Suppose X is compact. Let (p_n) be a Cauchy sequence in X. We know there is a convergent subsequence (p_{n_i}). Let p be its limit and let $r > 0$. Then there exists $N, M \in J$ such that

$$|p_{n_i} - p| < r/2 \text{ and } |p_n - p_m| < r/2$$

for $i > N$ and $n, m > M$. Let $L = \max M, N$. As $n_i \geq i$, if $n > L$ and $i > L$,

$$|p_n - p_{n_i}| < r/2$$

and also $|p_{n_i} - p| < r/2$.

Corollary 1.2 *In $\mathbf{R}^k$, every Cauchy sequence converges.*

Theorem 1.3 *Every bounded monotonic sequence of real numbers converges.*

A metric space in which every Cauchy sewuence converges is said to be **complete**.

2 Upper and Lower limits

Let $b := (b_i)_i$ be a sequence of real numbers.

We say (b_i) goes to $+\infty$ if for every integer N $b_i > N$ for sufficiently large i. This is written,

$$b_i \to +\infty.$$

$b_i \to -\infty$ is defined similarly. Monotonically increasing or decreasing sequences have limits in the extended real numbers.

We define $\limsup (b_i)$ to be the limit in the extended real numbers of the monotonically decreasing sequence (e_n) where

$$e_n = \sup\{b_i : i \geq n\}.$$

We also call this b^*.

Alternatively, let E be the set of limits of subsequences of b. Then,

$$s^* = \sup E.$$

Proof. We'll do the case $s^* \neq \infty$. In this case, $\{b_i\}$ is bounded.

For each integer $m > 0$ there exists an integer N so that for all $n > N$, $b^* - (1/m) < e_n \leq b^*$. This means we can pick an $k_1 < k_2 < \cdots$ such that

$$b^* - (1/i) < b_{k_i} \leq b^*.$$

Thus (b_{k_i}) is a subsequence of (b) and $\lim_{n\to\infty} b_{k_i} = b^*$
and so $\sup E \geq b^*$.

Now suppose (b_{i_n}) is a convergent subsequence of b and $e \in E$ is its limit. Then

$$e \leq \sup\{b_{i_n} \colon n \geq m\} \leq e_{i_m}.$$

It follows that $e \leq b^*$.

The $\liminf$ or b_* is defined in a smilar way.

Examples.

i) $b := (1, -1, 1, -1, \ldots)$.

ii) Suppose b is a sequence containing all rational numbers.

iii) Let $b_n = \sqrt[n]{n}$.

3 Series

If $(a_n)_n$ is a sequence of complex numbers.

$$\sum_{i=1}^{m} a_i := a_1 + a_2 + \cdots + a_m$$

and we call $\sum_{i=1}^{\infty} a_i$ a **series** and set it equal to

$$\lim_{m\to\infty} \left(\sum_{i=1}^{m} a_i\right)$$

if this converges. If the limit exists the series is said to converge otherwise it is said to diverge.

Theorem 3.1 *The series $\sum a_n$ converges if and only if for all ϵ there exists an integer N such for $m \geq n \geq N$*

$$\left|\sum_{i=n}^{m} a_i\right| \leq \epsilon.$$

Do the following exercises in Chapter 3: 21 and 22. Also do

A. (a) Is every convergent sequence in a metric space Cauchy. Prove your answer. (b) Show the sequence $a_k = \sum_{n=1}^{k} \frac{1}{n}$ is not Cauchy.