Real Analysis

Lecture 11

1 Convergent Sequences

A sequence $(p_n)_{n\in J}$ in X is said to **converge** in X if there exists a $p \in X$ such that for each $r > 0 \in \mathbf{R}$ there exists an $m \in J$ such that

$$p_n \in N_r(p)$$

for all $n \geq m$ and in this case we say (p_n) converges to p. Otherwise (p_n) is said to **diverge**.
Examples.

Some basic facts:
(i) A convergent sequence can only converge to one point.

(ii) A convergent sequence is bounded.

(iii) If $E \subseteq X$ then $p \in E'$ if and only if there exists a sequence $(p_n)_{n\in J}$ of distinct elements of E which converges to p.

2 Sequences in C and $\mathbf{R}^k$

Basically, what you expect to be true is true.

Theorem 2.1 *Suppose $(s_n)_{n\in J}$ s converges to s in $\mathbf{C}$, $s_n \neq 0$ and $s \neq 0$. Then $1/s_n$ converges to $1/s$.*

Proof. We know for all $r > 0$, $|s_n - s| < r$ for large n. Then,

$$\left|\frac{1}{s} - \frac{1}{s_n}\right| < \frac{r}{|ss_n|} \qquad (*)$$

for large n. Also, for large n, $|s_n| > \frac{1}{2}|s|$ so

$$\frac{1}{|s_n|} < \frac{2}{|s|}.$$

Combining this with $(*)$ we get

$$\left|\frac{1}{s} - \frac{1}{s_n}\right| < \frac{2r}{|s|^2}.$$

What are some other limits one might expect?

Theorem 2.2 *A sequence $(\mathbf{x}_n)$ in $\mathbf{R}^k$ where $\mathbf{x}_n = (x_{n\,1}, \ldots, x_{n\,k})$ converges to $\mathbf{x} = (x_1, \ldots, x_k)$ in $\mathbf{R}^k$ if and only if $(x_{n,i})_n$ converges to x_i for all i.*

3 Subsequences

If $S = (a_n)_n$ is a sequence in X, then a subsequence is a sequence $T = (b_n)_n$ where $b_i = a_{n_i}$ for some $n_i \in J$ and $n_{i+1} > n_i$.
Example and Non-example. $(2, 1, 4, 3, 6, 5, \ldots)$ is not a subsequence of $(1, 2, 3, 4, 5, 5, \ldots)$ although $(2, 4, 6, \ldots)$ is.

Theorem 3.1 *Every sequence in a compact metric space has a convergent subsequence.*

Proof.

Corollary 3.2 *Every bounded sequence in $\mathbf{R}^k$ has a convergent subsequence.*

4 Cauchy Sequences

A sequence $(p_n)_{n\in J}$ in X is said to be **Cauchy** if for every $r > 0 \in \mathbf{R}$, there exists an $N \in J$ such that

$$|p_n - p_m| < r$$

for all $n, m > N$.
Non-Example. $a_k = \sum_{n=1}^{k} \frac{1}{n}$.

Theorem 4.1 *Every Cauchy sequence in a compact metric space converges to a point of X.*

5 Homework

Read 52b-55b. Do the following exercises in Chapter 3:
14, 16. Also
A. Show a finite union of compact sets in a metric space is compact.
B. Show that a k-cell is not the union of two disjoint open sets.