Linear Algebra

Midterm Problems

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Set 3 (Last Set)

In the following V will be a finite dimensional vector space over $\mathbf{F}$.

21. Find all the linear maps from $\mathbf{R}^3$ to $\mathbf{R}$ whose null spaces are spanned by $(1, -1, 0)$ and $(4, 2, 1)$.

22. Let V and W be two vector spaces over $\mathbf{F}$. Let S be a basis for V and α be a map from S to W. Show that there exists a unique linear map $L: V \to W$ such that

$$L(s) = \alpha(s)$$

for $s \in S$.

23. True-False If false give a counter-example. Suppose W is an inner product space over $\mathbf{F}$ and T is an operator on W.

(i) If L is a linear map from $\mathbf{R}^5$ to $\mathbf{R}^3$, the null space of L has dimension 2.

(ii) If S is an operator on W, $(TS)^* = T^*S^*$.

(iii) If T is an invertible operator on W so is T^*.

(iv) If a is an eigenvalue of T, a is an eigenvalue of T^*.

(v) If $f(x) \in \mathbf{F}[x]$ every T-invariant subspace is also $f(T)$-invariant.

(vi) An infinite dimensional inner product space can't have an orthonormal basis.

(vii) If T is injective it is an isomorphism.

24. Suppose U, W are subspaces of V. Show there exists a operator on V which takes U onto W if and only if $\dim U \geq \dim W$.

25. Suppose A is a linear operator of V. Suppose v_1, v_2 are two non-zero vectors in V and a_1, a_2 are two distinct elements of F such that $Av_i = a_i v_i$. Show v_1 and v_2 are independent.

26. Find $f \in \mathcal{P}_3(\mathbf{R})$. such that $f'(0) = f''(0) = 0$ and

$$\int_0^1 |1 + x^2 - f(x)|^2 \mathrm{d}x$$

is as small as possible.

27. If $v_1, \ldots, v_n$ is a basis of V there exists a unique inner product on V so that this is an orthonormal basis.

28. Suppose $v_1, \ldots, v_n$ is a basis of V. Let $\hat{v}_i$ be the linear functional which takes v_j to $\delta_{i\,j} := \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$. Show $\hat{v}_1, \ldots, \hat{v}_n$ is a basis for $\hat{V}$.