Linear Algebra

Midterm Problems

Robert F. Coleman

In the following W will be a vector space (not necessarily of finite dim.), V will be a finite dimensional vector space over $\mathbf{F}$ and A will be a linear operator on V.

Set 1

1. True-False. If false explain why (e.g. give a counter-example).

(i) Any surjective linear map from a finite dimensional vector space to itself is an isomorphism.

(ii) The vector spaces over $\mathbf{R}$, $\mathbf{R}^3$ and the space of polynomials over $\mathbf{R}$ of degree at most 3 are isomorphic.

(iii) If S is a linearly dependent subset of W then every element of S is a linear combination of the other elements of S.

(iv) If $v \in V$ is an eigenvector of A with eigenvalue 0 then $\operatorname{range} A \neq V$.

(v) V contains an A-invariant subspace.

(vi) The image of A is A-invariant.

(vii) If L is an operator on V and $\lambda \in \mathbf{F}$, the set of vectors in V such that $Lv = \lambda v$ is L-invariant.

(viii) If A is a diagonalizable linear operator on V and W is an A-invariant subspace then A_W is diagonalizable.

(vix) If A is a linear operator such that Av is a multiple of v for all v in V, then A is a scalar.

(x) If a linear operator has an upper triangular matrix with respect to some basis then it has a basis of eigenvectors if and only if this matrix is diagonal.

2. Let x_i for $1 \leq i \leq 5$ be the linear map from $\mathbf{R}^5$ to $\mathbf{R}$ which takes $(a_1, ..., a_5)$ to a_i. Find a basis for the subspace of $V = \mathbf{R}^5$ in the null space of both linear maps $(x_1, \ldots, x_5) \mapsto x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5$ and $(x_1, \ldots, x_5) \mapsto x_1 + 2x_2$.

3. Suppose a linear operator A on V has a matrix of the form

$$\begin{pmatrix} X & Y \\ 0 & Z \end{pmatrix}$$

with respect to some basis of V, where X, Y, Z are $m \times m$, $m \times (n-m)$ and $(n-m) \times (n-m)$ matrices respectively . Show V has an m-dimensional subspace that is A-invariant.

4. Suppose V is two-dimensional. Show that if A has at least three distinct one dimensional invariant subspaces, then A is multiplication by a scalar.

5. If A and B are linear operators on V and $\operatorname{null}(A) \supseteq \operatorname{null}(B)$, show there exists a unique linear map C: $\operatorname{range}(B) \to V$ such that $C(B(v)) = A(v)$.

6. Suppose S and T are operators on V and S is invertible. Show T and STS^{-1} have the same eigenvalues.

7. Let L be a linear operator on $V = \mathbf{R}^4$ such that the matrix of L is

$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

with respect to the standard basis. Determine the L-invariant subspaces of V.

8. Suppose $\mathbf{F} = \mathbf{C}$, A is a linear operator on V. Show V contains a k-dimensional A-invariant subspace for $k \leq n$.

9. Show the vectors $(1, 0, \ldots), \ldots, (0, \ldots, 0, 1, 0, \ldots), \ldots$ do not make up a basis for $\mathbf{F}^\infty$.

10. If V has dimension n over $\mathbf{F}$, show the dimension of the space of linear operators on V has dimension n^2.

11. Let T be a map from V to $\mathbf{F}^m$. Prove that T is linear if and only if there exists linear maps $L_1, \ldots, L_m$ from V to $\mathbf{F}$ such that

$$T(x) = (L_1(x), \ldots, L_m(x)).$$