Linear Algebra

Final Problems III

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Hint for 4: If c is a complex number $a = (c + \bar{c})/2$ and $b = -(a - \bar{b})/2$ are real and $c = a + bi$.

Hint for 7: Extend bases for W_1 and W_2 to bases for V_1 and V_2.

21 (simplified). Assume $\mathbf{F} = \mathbf{C}$ and A has only one eigenvalue. (Hint: $X \mapsto AX - XA$ is a linear map. Use Jordan form.)

29. Suppose 0 is the only eigenvalue for N. Show N is nilpotent if $\mathbf{F} = \mathbf{C}$ but not necessarily if $\mathbf{F} = \mathbf{R}$.

30. Give an example of an operator over $\mathbf{C}^4$ whose characteristic polynomial equals $(x - 7)^2(x - 8)^2$ and whose minimal polynomial equals $(x - 7)(x - 8)^2$.

31. Suppose M is a matrix in Jordan form for an operator A on an n-dimensional vector space V with only one complex number on the diagonal, with respect to a basis $v_1, \ldots, v_n$. Let $1 = i_0 < i_1 < \cdots < i_r \leq n$ be the integers i such that either $i = 1$ or M has a zero in the $(i-1, i)$-th place. Show $\{v_{i_0}, \ldots, v_{i_r}\}$ is a basis for the unique eigenspace for A.

32. True-False. If false explain why.

(i) If $AB = BA$ and A has n distinct eigenvalues then B has diagonal matrix with respect to some basis.

(ii) If $A, B \in \mathcal{L}(V)$, $\operatorname{char}_{AB}(x) = \operatorname{char}_A(x) \operatorname{char}_B(x)$.

(iii) The sum of two normal operators is normal.

(iv) An operator with an upper triangular matrix with only one value on the diagonal has a basis of eigenvectors if and only if the matrix is diagonal.

(v) Positive operators have cube roots.

(vi) The product of two positive operators is positive.

(vii) The only positive isometry is the identity.

(viii) The Jordan form of an operator on $\mathbf{F}^2$ is unique.

(ix) If $f(x)$ is a polynomial over $\mathbf{F}$ such that $f(0) \neq 0$ and L is an operator on V such that $f(L) = 0$, then L is invertible.