Linear Algebra

Robert F. Coleman

Final Probems II

In the following V will an n-dimensional space over a field $\mathbf{F}$ and L is a linear transformation on V over K. Also, for a vector space W over $\mathbf{F}$, $\hat{W} = \mathcal{L}(W, \mathbf{F})$.

9. Suppose W is a subspace of V. Let U be the subspace of $\hat{V}$ of linear functionals which vanish on W. Show the map $\hat{V} \to \hat{W}$ which takes a linear functional to its restriction to W is surjective with null space U.

10. Suppose $\mathbf{F} = \mathbf{R}$ and T is a normal operator on $\mathbf{F}^2$. Use the fact that $T + T^*$ is self-adjoint to show that either $T = T^*$ or $T = trace(T)I - T^*$.

11. True-False. If false, explain why.

(i) If L has n distinct eigenvalues then there exists a basis B such that $[L]_B$ is diagonal.

(ii) If $V = U \oplus W$ and S_1 spans U and S_2 spans W then $S_1 \cup S_2$ spans V.

(iii) $\hat{\hat{\hat{V}}}$ is isomorphic to $\hat{V}$.

(iv) If two operators have distinct Jordan forms they are not conjugate.

(v) If $V = U \oplus W$ and S_1 spans U and S_2 spans W then $S_1 \cup\ S_2$ spans V.

(vi) $\mathbf{F}^4$ is isomorphic to the vector space of polynomials over $\mathbf{F}$ of degree at most 3.

(vii) There do not exist linear functionals on $\mathbf{F}^3$ whose null spaces are isomorphic to $\mathbf{F}^2$.

(viii) Suppose B is a finite basis of V and T, U are linear transformations of V such that $[T]_B = [U]_B$. Then $T = U$.

(ix) If T is a linear transformation of V such that $T^2 = I_V$, then $T = I_V$ or $T = -I_V$.

(x) Let B be a basis for V. Then the subset $\{\hat{b} : b \in B\}$ is a basis for $\{\hat{V}\}$.

13. Suppose W is a subspace of V. Show there exists a projection of V onto W.

14. Suppose A is a linear transformation of V. Suppose v_1, v_2 are two non-zero vectors in V and a_1, a_2 are two distinct elements of F such that $Av_i = a_i v_i$. Show v_1 and v_2 are independent.

15. Let $T \colon F^n \to F$ be a linear functional. Show that there exist elements $a_1, ..., a_n$ in F such that $T(x_1, ..., x_n) = \sum_{i=1}^{n} a_i x_i$.

16. Suppose T is positive operator on V. Show T is invertible if and only if $\langle Tv, v\rangle > 0$ for every non-zero v.

17. Suppose $A \colon V \to U$ is surjective and $\operatorname{Null} A$ is L-invariant. Show there exists a unique operator T on U such that

$$T(A(v)) = A(L(v)).$$

18. Let A, L and T be as in the previous problem. Suppose the restriction of L to $\operatorname{Null} A$ and T are invertible. Show L is invertible.
A-invariant subspaces of $\mathbf{F}^2$.

19. Suppose $\mathbf{F} = \mathbf{R}$ and $T \colon V \to V$ is self-adjoint, $\lambda \in \mathbf{R}$, $\epsilon > 0$ and there exists a non-zero vector $v \in V$ such that

$$||Tv - \lambda v|| < \epsilon ||v||$$

Show T has an eigenvalue $\lambda' \in \mathbf{R}$ with $|\lambda' - \lambda| < \epsilon$.

20. Let $A \in \mathcal{L}(V)$. Show the set of operators of the form $AX - XA$, $X \in \mathcal{L}(V)$ is a vector space of dimension at most $n^2 - n$.

21. Suppose L is the operator on $\mathbf{R}^3$, $L \colon (a, b, c) \mapsto (b, c, 0)$. Write $L = S\sqrt{L^*L}$ where S is an isometry. Determine how many choices there are for S.

22. Let $F = \mathbf{R}$ and $V = \mathbf{R^3}$. Compute the matrix of the linear transformation: $(x, y, z) \mapsto (z, 4x + y, 2x - y + z)$. Is this transformation invertible?