# Linear Algebra

Lecture CR

Robert F. Coleman

## 1. Normal Operators on real IP Spaces

**Theorem.** *Suppose $T$ is a normal operator on a real inner product space $(V, \langle\ ,\ \rangle)$. There exist mutually perpendicular $T$-invariant subspaces $W_i$ of $V$ of dimension at most 2 such that*

$$V = W_1 \oplus W_2 \oplus \cdots \oplus W_m.$$

Lets deduce this from the CST.

**CST.** *Let $L$ be a linear operator on a complex inner product space $U$. Then $U$ has an orthonormal basis of eigenvectors for $L$ if and only if $L$ is normal.*

or equivlently, let $\lambda_1, \ldots, \lambda_k$ be tthe distinct eigenvalues of $L$. Then if $U_{\lambda_i}$ is the $\lambda_i$ eigenspace, $U_{\lambda_j}$ is perpendicular to $U_{\lambda_i}$ if $i \neq j$ and

$$U = U_{\lambda_1} \oplus U_{\lambda_2} \oplus \cdots \oplus U_{\lambda_k}.$$

Now let $U = V_{\mathbf{C}}$ with inner product $\langle\ ,\ \rangle_{\mathbf{C}}$ and $L = T_{\mathbf{C}}$. Then $L$ is normal and we can apply the CST. First we note that

$$\overline{U_{\lambda_i}} = U_{\bar{\lambda}_i}.$$

So if $\lambda_i \in \mathbf{R}$ we can find an ON basis for $U_{\lambda_i}$ in $V$ ($\dim_R(U_{\lambda_i} \cap V) = \dim_{\mathbf{C}} U_{\lambda_i}$ because $\overline{U_{\lambda_i}} = U_{\lambda_i}$). Let $v_1, \ldots, v_r$ be the union of all these orthonormal bases we've found for $U_{\lambda_i}$ when $\lambda_i \in \mathbf{R}$. They are perpendicular eigenvectors and $r$ is the dimension of

$$\bigoplus_{\lambda_i \in \mathbf{R}} U_{\lambda_i}$$

Suppose $i_1, \ldots, i_s$ is a subset of $\{1, \ldots, k\}$ so that $\lambda_{i_1}, \bar{\lambda}_{i_1}, \ldots, \lambda_{i_s}, \bar{\lambda}_{i_s}$ are the distinct non-real eigenvalues for $L$. Let $u_1, \ldots, u_t$ be a union of orthonormal eigenbases for the $U_{\lambda_{i_j}}$ for $j \leq s$. Then $\dim V = r + 2t$. Thus

Let $U_i = \mathbf{C}u_i + \mathbf{C}\bar{u}_i$, then these $U_i$ are perpendicular, of dimension 2 and

$$\overline{U}_i = U_i.$$

Thus $\dim(U_i)_0 = 2$ and so we can take $m = r + t$

$$W_i = \begin{cases} \mathbf{R}v_i & i \leq r \\ (U_{i-r})_0 & r \leq i \leq r + t \end{cases}.$$ ∎

Problems. A. Prove the above equivolence. B. Prove $V = \bigoplus_{i=1}^{m} W_i$.