Linear Algebra

Lecture 7

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1. Linear Maps

Suppose V and W are vector spaces over $\mathbf{F}$. A linear map f from V to W is a function $f\colon V \to W$ that preserves linear structure, i.e. is such that $f(av + bw) = af(v) + bf(w)$.

If $v_1, \ldots, v_n$ is a basis of V and $w_1, \ldots, w_n$ are elements of W, there is a unique linear map L from V to W such that $L(v_i) = w_i$ and every linear map from V to W is of this form.

Let $\mathcal{L}(V, W)$ be the set of all linear maps from V to W. This set is naturally a vector space over $\mathbf{F}$.

Finally, if U is a third vector space, $T \in \mathcal{L}(V, W)$ and $S \in L(W, U)$ then $S \circ T \in \mathcal{L}(V, U)$.

2. Null Space and Range

Suppose L is a linear map from V to W. We say L is an **isomorphism** if and only if L is one-to-one and onto. Then the **null space** of L denoted $\operatorname{null} L$ is the set

$$\{v \in V \colon Lv = 0\}.$$

The **range** of L, denoted $\operatorname{range} L$, is the set

$$\{Lv \colon v \in V\}.$$

Proposition. *L is an isomorphism if and only if null $(L) = 0$ and range $(L) = W$.*

Proof.

Lemma. *Suppose U and V are subspaces of W then the mao $(u, v) \to u + v$ from $U \oplus V \to W$ is linear and an isomorphism if and only if W is the direct sum of U and V in Axler's sense.*

Proof.

Propostion. *null L is a subspace of V and range L is a subspace of W.*

Proposition. *Suppose $L \colon V \to W$ is a linear map. Then L is injective if and only if null $L = \{0\}$.*

Rank plus Nullity Theorem. *Suppose V is finite dimensional and $L \colon V \to W$ is a linear map. Then range L is finite dimensional and*

$$\dim V = \dim(\operatorname{null} L) + \dim(\operatorname{range} L).$$

For next time: Read pages 47-52. Do problems 1, 3 and 5 in Chapter 3.