

# Linear Algebra

## Lecture 5

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### Bases and Dimension

**Theorem.** *If  $v_1, \dots, v_n$  are independent vectors in  $V$  and  $W =: \text{Span}\{v_1, \dots, v_n\}$*

*(i) no subset of  $W$  with fewer than  $n$  elements spans  $W$  and (ii) no subset of  $W$  with more than  $n$  elements is independent.*

*Proof.* Suppose  $m < n$  and  $\text{Span}\{w_1, \dots, w_m\} = W$ .

Proof of (ii).

**Corollary.** *If  $V$  is spanned by  $m$  independent vectors, any linearly independent subset of  $V$  has fewer than  $m$  elements.*

**Lemma.** *Suppose  $v_1, \dots, v_n$  are linearly independent elements of  $V$  and  $w \in V$  is not in  $\text{span}\{v_1, \dots, v_n\}$ . Then  $v_1, \dots, v_n, w$  are independent.*

*Proof.*

A set of vectors  $S$  in  $V$  is a **basis** if and only if the elements in  $S$  are independent and  $\text{span}S = V$ . When  $S$  is finite and has  $n$  elements we say  $V$  has **dimension**  $n$ .

From what we know about linear independence,  $v_1, \dots, v_n$  is a basis of  $V$  if and only if there is one and only one way to write an element of  $V$  as a linear combination of the  $v_i$ .

Examples:

*Questions*

- A. Why did you give us a different definition of direct sum than Axler did?
  
  
  
  
  
  
  
  
  
  
- B. Is there a geometric definition of span?
  
  
  
  
  
  
  
  
  
  
- C. Can an infinite set of vectors be independent?
  
  
  
  
  
  
  
  
  
  
- D. Is the set of polynomials of degree  $n$  a subspace of  $\mathbf{F}[T]$ ?

*Exercises*

- 1. If  $v_1, \dots, v_n$  spans  $V$  so does  $v_1 - v_2, \dots, v_{n-1} - v_n, v_n$ .
  
  
  
  
  
  
  
  
  
  
- 2. If  $v_1, \dots, v_n$  are independent so are  $v_1 - v_2, \dots, v_{n-1} - v_n, v_n$ .
  
  
  
  
  
  
  
  
  
  
- 3. Suppose  $v_1, \dots, v_n$  are linear independent. Then if  $\{v_1 + w, \dots, v_n + w\}$  are linear dependent  $w \in \text{span}\{v_1, \dots, v_n\}$ . Give an example.
  
  
  
  
  
  
  
  
  
  
- 6. The vector space of continuous functions on  $[0, 1]$  is infinite dimensional.

7.  $V$  is infinite dimensional iff there exists an infinite set of independent vectors.

10. Suppose  $V$  has dimension  $n$ . Show there exist one-dimensional subspaces  $U_i$  such that

$$V = U_1 \oplus U_2 \oplus \cdots \oplus U_n$$

in Axler's sense.

For next time: Read pages 37-47. Do problems 8, 11, 12 in Chapter 2.