Linear Algebra

Lecture 4

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1. Linear Independence

Suppose $v_1, \ldots, v_n$ are n vectors in V. Then, $v_1, \ldots, v_n$ are said to be **linearly dependent** if there exist $a_1, \ldots, a_n \in \mathbf{F}$ not all zero such that

$$a_1 v_1 + \cdots + a_n v_n = 0$$

and **linearly independent** otherwise. Another way to say this is,

Lemma. *(almost) The vectors $v_1, \ldots, v_n$ are linear independent if and only if there is only one way to write an element in Span$\{v_1, \ldots, v_n\}$ in the form*

$$a_1 v_1 + \cdots + a_n v_n, \quad a_i \in \mathbf{F}.$$

Proof.

Examples. (i) $\mathbf{F}^n$

(ii) Polynomials.

A useful way to think about this is $v_1, \ldots, v_n$ are linearly dependent and $n > 1$ if and only if one of the v_j, is a linear combination of the others. I.e., $\text{span}\{v_1, \ldots, v_n\} = \text{span}\{v_1 \ldots, \hat{v}_j, \ldots, v_n\}$

Theorem. *If* $v_1, \ldots, v_n$ *are independent vectors in* V *and* $W =: Span\{v_1, \ldots, v_n\}$ *(i) no subset of* W *with with fewer than* n *elements spans* W *and (ii) no subset of* W *with more than* n *elements is independent.*

Proof. Suppose $m < n$ and $Span\{w_1, \ldots, w_m\} = W$, then $v_1 = a_1 w_1 + a_m w_m$ for some $a_i \in \mathbf{F}$.

Proof of (ii).

Corollary. *If* V *is spanned by* m *independent vectors, any linearly independent subset of* V *has fewer than* m *elements.*

For next time: Read pages 27m-34. Do problems 6, 7 and 10 in Chapter 2..