

Linear Algebra

Lecture 32

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1. Toward Jordan Form

Suppose V is a vector space over \mathbf{F} .

Theorem. If $\mathbf{F} = \mathbf{C}$ and $\lambda_1, \dots, \lambda_d$ are the eigenvalues of T , then $V = \bigoplus_i V(\lambda_i)$.

We say an operator N on a vector space V is **cyclic** if there exists an element $w \in V$ such that V is the span of $w, N(w), \dots, N^k(w), \dots$. Call such a w a **generator**.

Lemma. If N is nilpotent, $N^d = 0$, $N^{d-1} \neq 0$ and N is cyclic on V , then V has dimension d and is spanned by $w, N(w), \dots, N^{d-1}(w)$, if w is a generator.

Proposition. If N is a nilpotent operator on W and $W = \bigoplus W_i$ where W_i is N -invariant and $N|_{W_i}$ is cyclic of dimension d_i , then

$$\text{Null } N^r = \sum_i N^{d_i-r} W_i,$$

where $N^{d_i-r} W_i = W_i$, if $d_i < r$.

Proof.

Theorem. If N is a nilpotent operator on V there exists N -invariant cyclic subspaces V_i of V such that

$$V = \bigoplus_i V_i.$$

Proof. Suppose $W = \bigoplus_{i=1}^k V_i \subset V$ with V_i cyclic of dimension d_i and if $v \notin W$, $N^{d_i} v = 0$ for all i . Let $d_{k+1} = \max \{d: \exists w \in V, N^{d-1} w \notin W, N^d(w) = 0\}$. Suppose

$w \in V$, $N^{d_{k+1}-1}(w) \notin W$, $N^{d_{k+1}}(w) = 0$. Claim: $W \cap V(w) = 0$. Suppose

$$a_1 w + \cdots + a_{d_{k+1}} N^{d_{k+1}-1} w \in W.$$

The set $\{d: \exists w \in W, N^{d-1} w \notin W, N^d(w) = 0\}$ is not empty. Suppose $N(w) \in W$, $w \notin W$. There exists $1 \leq d \leq d_i$ and $N^d(w) = 0$. Then $N^{d-1}(N(w)) = 0$ so

$$N(w) \in \sum_i N^{d_i-(d-1)} V_i \subseteq N(W).$$

Thus there exists $y \in W$, $N^{d_1-(d-1)} y = N(w)$ so

$$N(w - N^{d_1-d} y) = 0.$$

Homework for Friday

Prove the above lemma. Let V be a vector space over \mathbf{F} . Let $W_n(V)$ be as in lecture 29's homework. Suppose M and N are in $\mathcal{L}(V)$. Suppose $d \in W_n(V)$. Let

$$N(d)(v_1, \dots, v_n) = d(N(v_1), \dots, N(v_n)).$$

Show $N(d) \in W_n(V)$, $d \mapsto N(d)$ is linear and $M(N(d)) = (M \circ N)(d)$.