Linear Algebra

Lecture 29

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1. Singular Value Decomposition.

The **singular values** of an operator T are the eigenvalues of $\sqrt{T^*T}$ counted with multiplicity. Write $T = S\sqrt{T^*T}$. Suppose $e_1, \ldots, e_n$ is an orthoganal eigenbasis for $\sqrt{T^*T}$ with eigenvalues $s_1, \ldots, s_n$ and $f_i = Se_i$. Then $f_1, \ldots, f_n$ is an orthogonal basis and

Theorem.

$$Tv = s_1\langle v, e_1\rangle f_1 + \cdots + s_n\langle v, e_n\rangle f_n.$$

Some uses of SVD: T is an isometry if and only if all its singular values equal 1. If $s_1 \leq s_i \leq s_n$ for all i, $s_l||v|| \leq ||Tv|| \leq s_g||v||$.

$$(s_1\langle v, e_1\rangle)^2 + \cdots + (s_n\langle v, e_n\rangle)^2$$

2. Generalized Eigenvectors

Suppose V is a vector space over $\mathbf{F}$ which doesn't necessarily have an inner product.

Let L be an operator on V. Then an eigenvector v is a vector in the kernel of $L - \lambda$ for some λ. What do we do if we cann't find an eigenbasis?

By a **generalized eigenvector** of eigenvalue λ for T we mean a vector v such that $(T-\lambda)^d v = 0$ for some d. We will show that if $\mathbf{F} = \mathbf{C}$ we can always find a basis of generalized eigenvectors and even better with respect to this basis the matrix for T is of the form $\begin{pmatrix} A_1 & 0 & \ldots & 0 \\ 0 & A_2 & \ldots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \ldots & \ldots & A_m \end{pmatrix}$, $\mathrm{Mat}(A_i) = \begin{pmatrix} \lambda_i & 1 & 0 & \ldots & 0 \\ 0 & \lambda_i & 1 & \ldots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \ldots & \ldots & \lambda_i & 1 \\ 0 & \ldots & \ldots & \ldots & \lambda_i \end{pmatrix}$.

Remark. *The eigenvalue of a non-zero generalized eigenvector is actually the eigenvalue of a real eigenvector.*

Lemma. *(i)* $\bigcup_i$ *Null* $(T-\lambda)^i =$ *Null* $(T-\lambda)^{\dim V}$ *and*
(ii) $\bigcap_i$ *Range* $T^i =$ *Range* $T^{\dim V}$.

Proof.

Lemma. *If L is an operator on V,*

$$V = L^{\dim V} V \oplus \mathit{Null}\, L^{\dim V}.$$

Proof.

Reading and problems for next time: Read pages 163-168. Do problems 32 and 33 in §7. and 3 and 4 in §8.

What are the singular values of

$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad ?$$

Let V be a vector space over $\mathbf{F}$ and $W_n(V)$ be the space of functions d into $\mathbf{F}$ on V^n such that

$$d(v_1, \ldots, v_{k+1}, v_k, \ldots, v_n) = -d(v_1, \ldots, v_n) \tag{1}$$

and

$$d(v_1, \ldots, au_i + bw_i, \ldots, v_n) = ad(v_1, \ldots, u_i, \ldots, v_n) + bd(v_1, \ldots, w_i, \ldots, v_n). \tag{2}$$

Show $W_n(V)$ is a vector space. What is its dimension if $n = \dim V = 2$?