Linear Algebra

Lecture 28

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1. Isometries

Isometries are operators T such that $||Tv|| = ||v||$.

Lemma. *Suppose T is an isometry. Then $TT^* = T^*T = I$. In particular, $T^{-1} = T^*$ and T is normal.*

Proof.

$$\langle v, v\rangle = \langle Tv, Tv\rangle = \langle T^*Tv, v\rangle$$

Corollary. *The columns of the matrtix of an operator T on V with respect to an orthonormal standard basis form an orthonormal basis of $\mathbf{F}^n$ if and only if T is an isometry.*

Proof.

2. Theorems about isometries

Theorem. *Every isometry of the plane $\mathbf{R}^2$ (with standard inner product) is either a flip or a rotation.*

Proof. Let T be a isometry of R^2. If T is not self-adoint,

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

$a^2 + b^2 = 1$.

Lemma. *An operator T is an isometry if and only if*

$$\langle Sv, Sw \rangle = \langle v, w \rangle$$

for all v and w.

Proposition. *If T is an isometry of $V = \mathbf{R}^n$, there exists mutually orthogonal T-invariant subspaces V_i V of dimension at most 2*

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_m$$

such that $T|_{V_i}$ is multipliction by ± 1 if $\dim V_i = 1$ and is a rotation otherwise.

In particular, if $n = 3$, T preserves a plane and the perpendicular line.

$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \stackrel{?}{=} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

Reading and problems for next time: Read pages 1 & 2 of hanout . Do problems 26 and 27 in §7.