Linear Algebra

Lecture 27

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1. Normal Operators on real IP Spaces

Theorem. *Suppose T is a normal operator on a real inner product space $(V, \langle\ ,\ \rangle)$. There exist mutually perpendicular T-invariant subspaces W_i of V of dimension at most 2 such that*

$$V = W_1 \oplus W_2 \oplus \cdots \oplus W_m.$$

Lets deduce this from the CST.

CST. *Let L be a linear operator on a complex inner product space U. Then U has an orthonormal basis of eigenvectors for L if and only if L is normal.*

Let $U = V_{\mathbf{C}}$ with inner product $\langle\ ,\ \rangle_{\mathbf{C}}$ and $L = T_{\mathbf{C}}$. Then L is normal we can apply the CST.

Exercise: $\overline{\beta v} = \bar{\beta}\bar{v}$ and $L(\bar{v}) = \overline{L(v)}$.

We can find an ON basis of eigenvectors $v_1, \ldots, v_n$, $u_1, \bar{u}_1, \ldots u_m, \bar{u}_m$ with eigenvalues $r_1, \ldots, r_n$, $c_1, \bar{c}_1, \ldots c_m, \bar{c}_m$ with $r_i \in \mathbf{R}$ and $c_i \notin \mathbf{R}$.

Let $V_i = \mathbf{C}v_i$ and $U_i = \mathbf{C}c_i + \mathbf{C}\bar{c}_i$. Then $\overline{V_i} = V_i$ and $\overline{U_i} = U_i$.

2. Isometries

Isometries are operators which preserve distance. Formally, T is an isometry if

$$||Tv|| = ||v||.$$

Examples.

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Facts about isometries:

The product of two isometries is an isometry.

Isometries are invertible.

The inverse of an isometry is an isometry.

If λ is an eigenvalue of an isometry $|\lambda| = 1$.

Lemma. *T is an isometry if and only if Then $TT^* = T^*T = I$. In particular, if T isss an isometry $T^{-1} = T^*$ and T is normal.*

Proof.

$$\langle v, v\rangle = \langle Tv, Tv\rangle = \langle T^*Tv, v\rangle$$

Corollary. *The columns of the matrtix of an operator T on V with respect to an orthonormal basis form an orthonormal basis of $\mathbf{F}^n$ if and only if T is an isometry.*

Homework for Friday

A. Suppose $(V, \langle\ ,\ \rangle_V)$ and $(W, \langle\ ,\ \rangle_W)$ are inner product spaces and $L \in \mathcal{L}(V, W)$. Show there exists a unique $L^a \in \mathcal{L}(W, V)$ such that $\langle L(v), w\rangle_W = \langle v, L^a(w)\rangle_V$. Do this explicitly when $V = \mathbf{R}$, $W = \mathbf{R}^2$ and $L(x) = (ax, by)$.

B. Suppose T is self-adjoint, $\lambda \in \mathbf{F}$ and $\epsilon > 0$. Suppose $||v|| = 1$ and $||Tv - \lambda v|| < \epsilon$. Then T has an eigenvalue λ' with $|\lambda' - \lambda| < \epsilon$. (Hint, let $e_1, \ldots, e_n$ be an ON basis with eigenvalues λ_i. Write $v = \sum a_i e_i$. Then $\sum |a_i|^2 = 1$ and $\sum |a_i(\lambda_i - \lambda)|^2 < \epsilon^2$.) Do the above exercise.