

Linear Algebra
Lecture 25
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Polar Decomposition

Let $A \in \mathcal{L}(V)$.

Lemma. *There exists a self-adjoint operator B on V with positive (real) eigenvalues such that $B^2 = A^*A$.*

Proof.

Theorem. *If A is invertible, AB^{-1} is an isometry.*

Proof.

$$\langle AB^{-1}v, AB^{-1}v \rangle =$$

Polar Decomposition. *If L is an operator on V , There exists an isometry T of V such that*

$$L = T\sqrt{L^*L}.$$

Example. Let $A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$. Then $A^* = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$ so

$$A^*A = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ -1 & 0 & 0 & 2 \end{pmatrix}$$

$$x^2 - 3x + 1, \frac{3 \pm \sqrt{5}}{2} = \left(\frac{1 \pm \sqrt{5}}{2}\right)^2. \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^2 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \text{ so}$$

$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

Linear Algebra Done Righter

Suppose $(V, \langle \cdot, \cdot \rangle)$ is a real inner product space. Let $V_{\mathbf{C}}$ be the set of symbols $\{v + iw : v, w \in V\}$. If $a, b \in \mathbf{R}$ and $u, v, w, y \in V$ set

$$(v + iw) + (u + iy) = (v + u) + i(w + y), (a + bi)(v + iw) = (av + iaw) + (-bw + ibv)$$

$$\text{and } \langle v + iw, u + iy \rangle_{\mathbf{C}} = \langle v, u \rangle + i\langle w, u \rangle - i\langle v, y \rangle + \langle w, y \rangle.$$

A. Show $(V_{\mathbf{C}}, \langle \cdot, \cdot \rangle_{\mathbf{C}})$ is a complex inner product space.

Identify V with $\{v + i0 : v \in V\} \subset V_{\mathbf{C}}$. Define $\overline{x + iy} = x - iy$ for $x, y \in V$ and if W is a subspace of $V_{\mathbf{C}}$, $\overline{W} = \{\overline{w} : w \in W\}$.

B. Show \overline{W} is a subspace of $V_{\mathbf{C}}$, $W_0 =: W \cap V$ is a subspace of V and if $\overline{W} = W$, $\dim_{\mathbf{R}} W_0 = \dim_{\mathbf{C}} W$ and $(W_0)_{\mathbf{C}} \cong W$.

If L is a linear operator on V set $L_{\mathbf{C}}(v + iw) = L(v) + iL(w)$.

C. Show if L is normal or self-adjoint, so is $L_{\mathbf{C}}$.

D. Deduce the RST from the CST.

Reading and problems for next time: Read pages 152-157. Do problems 9, 21, 22 and 23 in §7.