Linear Algebra

Lecture 24

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1. Invariant Subspaces

Proposition. *Let V_i be the λ_i-eigenspace. Suppose $V = \bigoplus_1^n V_i$. Then the invariant subspaces of V are $W_1 + W_2 + \cdots + W_n$ where $W_i \subseteq V_i$.*

Proof. Suppose W is L-invariant. Let $W_i = V_i \cap W$. Let $w \in W$. We can write $w = v_1 + \cdots + v_n$.

2. Two dimensional Inner Product Spaces over R

Suppose $\mathbf{F} = \mathbf{R}$, and $V = \mathbf{R}^2$ with standard inner product. Let T be an operator on V. Then we know T is self-adjoint if and only if its matrix with respect to any orthonormal basis is equal to its transpose. When is it normal?

Lemma. *The operator T is normal but not self-adjoint if and only if the matrix of T with respect to some orthonormal basis has the form*

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \qquad \text{with } b > 0$$

Proof. We need $||Tv|| = ||T^*v||$.

3. Normal Operators on real IP Spaces

Normal Theorem. *Suppose T is a normal operator on a real inner product space V. There exist mutually perpendicular T-invariant subspaces W_i of V of dimension* *at most* 2 *such that*

$$V = W_1 \oplus W_2 \oplus \cdots \oplus W_m.$$

In matrixland this means there is an orthonormal basis such that the matrix for T is of the form

$$\begin{pmatrix} A_1 & 0 & \dots & 0 \\ 0 & A_2 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & A_m \end{pmatrix}$$

where A_i is 1×1 or 2×2. In fact, we can ensure that if A_i is 2×2, it has the form

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

with $b > 0$.

Suppose T is a normal operator on $(V,)$. We want to break V down into one or two-dimension mutually perpendicular T-invariant subspaces. For this we need,

Theorem. *Suppose U is T-invariant. Then so is $U^\perp$.*

Proof of theorem.

$$0 = ||(T - \lambda)v|| = ||(T^* - \bar{\lambda})v||$$

Reading and problems for next time: Read pages 147-152. Do problems 17, 18, and 20 in §7.

Geometric Interpretation of Adjoints: Suppoise $(V, \langle\ ,\ \rangle)$ is an inner product space. Define $\langle\ ,\ \rangle_2$ on $V^2 = V \oplus V$ by $\langle (u, v), (w, z) \rangle_2 = \langle u, w \rangle + \langle v, z \rangle$. For a linear operator L on V let $G(L)$ be the "graph" of L, $\{(v, L(v)) : v \in V\}$. Show $G(L)^\perp = \{(-w, v) : (v, w) \in G(L^*)\}$.