Linear Algebra

Lecture 23

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1. Complex Spectral Theorem

Theorem. *Let L be a linear operator on an inner product space V. Then V has an orthonormal basis of eigenvectors for L with real eigenvalues if and only if L is self-adjoint.*

An operator T is **normal** if and only if $TT^* = T^*T$.

Example.

$$R(\theta) =: \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}.$$

$R(\theta)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos(\theta)x - \sin(\theta)y \\ \sin(\theta)x + \cos(\theta)y \end{pmatrix}$. Let $e(\theta) = \cos(\theta) + i\sin(\theta)$. Then

$$e(\theta)(x + iy) =$$

Theorem. *Let L be a linear operator on a complex inner product space V. Then V has an orthonormal basis of eigenvectors for L if and only if L is normal.*

Our Example.

$$R(\theta)\begin{pmatrix} 1 \\ i \end{pmatrix} =$$

Key Lemma. *An operator L on V is normal if and only if $||L(v)|| = ||L^*(v)||$ for all $v \in V$.*

Proof.

2. Polar Decomposition

Fact. *If $c \in \mathbf{C}$, there exists a complex number θ with $\theta\bar{\theta} = 1$ such that*

$$c = \theta\sqrt{c\bar{c}}.$$

Let L be any operator on V. Then L^*L is self-adjoint, but more is true.

Proposition. *There exists an unique self-adjoint operator S on V with non-negative eigenvalues such that $S^2 = L^*L$.*

Denote this operator $\sqrt{L^*L}$. We will prove,

Polar Decomposition. *There exists a operator T on V such that $T^*T = I$ and*

$$L = T\sqrt{L^*L}.$$

Reading and problems for next time: Read pages 144-147. Do problems 7, 13, and 16 in §7.