

# Linear Algebra

## Lecture 22

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### 1. Proof of the Real Spectral Theorem

Let  $(V, \langle \cdot, \cdot \rangle)$  be a finite dimensional inner product space over  $\mathbf{F}$ . (Don't assume  $\mathbf{F} = \mathbf{R}$ .)

**Lemma.** *If  $L$  is a self-adjoint linear operator on  $V$  and  $U$  is an  $L$ -invariant subspace, then so is  $U^\perp$ .*

*Proof.*

**Theorem.** *Let  $L$  be a linear operator on an inner product space  $V$ . Then  $V$  has an orthonormal basis of  $L$ -eigenvectors with real eigenvalues if and only if  $L$  is self-adjoint.*

*Proof.* First suppose  $L$  has an orthonormal basis of eigenvectors.

Now suppose  $L$  is self-adjoint. Proof by induction on the dimension of  $V$ :

We know self-adjoint operators have eigenvectors and every eigenvalue of a self-adjoint operator is real.

## Exercices

30.  $T$  is injective if and only if  $T^*$  is surjective.

31.  $T \in \mathcal{L}(V, W)$ .  $\dim \text{Null } T^* = \dim \text{Null } T + \dim W - \dim V$ .

32. Row rank = column rank.

1.  $\langle p, q \rangle = \int_0^1 p(x)q(x)dx$ . Let  $T(a_0 + a_1x + a_2x^2) = a_1x$ . Why isn't  $T$  self adjoint?

2. Is the product of two self-adjoint matrices self-adjoint?

3. Is the set of self-adjoint matrices a subspace?

**Reading and problems for next time:** Read pages 138-140(top). Do problems 4, 6, 15 in §7. (Hint for 6: Use Prop. 7.6 to show  $\text{Null } T = \text{Null } T^*$ , and Prop.

7.6.) Also, let  $T$  be the linear operator on  $V = \mathbf{R}^3$  whose matrix is  $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ ,

with respect to the standard basis. Find an orthonormal basis of  $V$  which consists of eigenvectors of  $T$ .