

Linear Algebra

Lecture 21

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1. The Midterm

1. (iii) If L is a linear map from \mathbf{R}^5 to \mathbf{R}^3 , the null space of L has dimension 2.
 - (iv) If T is a linear operator on V such that $T^2 = I_V$, (the identity operator of V) then $T = I_V$ or $T = -I_V$.
 - (viii) The product of two projections is a projection.
 - (ix) If a is an eigenvalue of T , a is an eigenvalue of T^* .
5. Suppose U, W are subspaces of V . Show there exists a operator on V which takes U onto W if and only if $\dim U \geq \dim W$.

2. Self-adjoint Operators

Let $(V, \langle \cdot, \cdot \rangle)$ be a finite dimensional inner product space.

An operator $T: V \rightarrow V$ is called self-adjoint if $\langle Tv, u \rangle = \langle v, Tu \rangle$.

Example. Suppose $V = \mathbf{R}^n$ with the standard inner product and L is an operator on V whose matrix M w.r.t. the standard basis is symmetric ($M = M^{trsp}$).

Real Spectral Theorem. *Let L be a linear operator on a real inner product space V . Then V has an orthonormal basis of eigenvectors for L if and only if L is self-adjoint.*

3. Down to work

Don't assume $\mathbf{F} = \mathbf{R}$.

Lemma. *Every eigenvalue of a self-adjoint operator is real.*

Proof.

Proposition. Suppose T is self-adjoint and

$$\langle Tv, v \rangle = 0$$

for all v . Then $T = 0$.

Proof.

$$\begin{aligned}\langle Tu, w \rangle + \overline{\langle Tu, w \rangle} &= \langle Tu, w \rangle + \langle Tw, u \rangle \\ &= \langle T(u + w), u + w \rangle - \langle Tu, u \rangle - \langle Tw, w \rangle.\end{aligned}$$

Lemma. Suppose T is self-adjoint $b, c \in \mathbf{R}$ and $b^2 - 4c < 0$. Then,

$Q(T) =: T^2 + bT + cI$ is invertible.

Proof.

$$\begin{aligned}Q(T) &= (T + b/2)^2 + (c - b^2/4) \\ \langle Q(T)v, v \rangle &= \langle (T + b/2)v, (T + b/2)v \rangle + (c - b^2/4)\langle v, v \rangle\end{aligned}$$

Corollary. Suppose T is self-adjoint. Then T has an eigenvalue.

Proof. Let $v \neq 0$ and $P(T)$ a non-zero polynomial of minimal degree such that $P(T)v = 0$. We can write

$$P(T) = \prod_{i=1}^r Q_i(T) \prod_{j=1}^s L_j(T),$$

where $Q_i(T)$ has degree 2 and no roots in \mathbf{F} and $L_j(T)$ has degree 1.

Reading and problems for next time: Read pages 133m-138. Do problems 1, 2 and 3 in §7.