

Linear Algebra

Lecture 20

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Review

We learned about vector spaces over a field and subspaces of a vector space. We know that if S is any subset of a vector space V , there is a smallest subspace of V containing S called the span of S .

A subset S of V is called a basis if every element of V can be expressed as a linear combination of elements of S in only one way. The number (maybe ∞) of elements in a basis for V is independent of the basis and is called the dimension of V .

$\{1, T, T^2, \dots\}$ is a basis for $\mathbf{F}[T]$.

A map L from one vector space V to another is said to be linear if it respects the vector space structures. Linear maps have null spaces and ranges and

$$\dim V = \dim \text{Null } L + \dim \text{Range } L.$$

If $L: V \rightarrow V$, a subspace U of V is said to be L -invariant if $L(U) \subset U$. We know if V is finite dimensional there exist L -invariant subspaces

$$0 = U_0 \subset U_1 \subset \dots \subset U_m = V$$

such that $\dim U_{i+1} - \dim U_i \in \{1, 2\}$. If it always equals 1, we can find an upper-triangular matrix for L . We can arrange this when $\mathbf{F} = \mathbf{C}$ and the numbers on the diagonal are the eigenvalues of L .

We've started learning how linear maps interact with inner products. We know that if $(V, \langle \cdot, \cdot \rangle)$ is a finite dimensional inner product space V possesses an orthonormal basis. If U is a subspace of V , we've defined the orthogonal projection of V on

U , P_U , which takes a vector in V to the vector in U closest to it. The map P_U is linear. We also know that if $L: V \rightarrow \mathbf{F}$ is a linear functional there exists a unique vector $v_L \in V$ such that

$$L(u) = \langle u, v_L \rangle, \quad \forall u \in V.$$

Is the map $L \mapsto v_L$ a linear map from $\hat{V} \rightarrow V$?

Questions

1. How can you tell whether you've found all invariant subspaces? What good are linear functionals?

2. Why can't all transformations over the reals be put in upper-triangular form? Isn't that what the row-echelon stuff does?

3. What good are eigenvectors? How many correspond to a given eigenvalue?

4. What is a norm?

Exercises

17. Suppose $P^2 = P$ and $\text{Null } P \subseteq (\text{Range } P)^\perp$. Show P is an orthogonal projection.

19. Show U is invariant under T if and only if $TP_U = P_UTP_U$.

21. In $U = \{(1, 1, 0, 0), (1, 1, 1, 2)\}$ find u such that $\|u - (1, 2, 3, 4)\|$ is minimal.

27. If $T(z_1, \dots, z_n) = (0, z_1, \dots, z_{n-1})$ what is T^* ?

28. Suppose λ is an eigenvalue for T . Show $\bar{\lambda}$ is an eigenvalue for T^* .

29. Show U is invariant under T if and only if U^\perp is invariant under T^* .

Reading and problems for next week: Read pages 127-133. Do problems 30, 31 and 32 in §6.