Linear Algebra

Lecture 19

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1. (viii) is true. 18 is false. Suppose $(V, \langle\ ,\ \rangle)$ is an inner product space over $\mathbf{F}$.

1. Linear Functionals

A linear functional on V is a linear map from V into $\mathbf{F}$. The set of all linear functionals on V is $\hat{V} := \mathcal{L}(V, \mathbf{F})$.

Theorem. *If F is a linear functional on V and $(v_1, \ldots, v_n)$ is an orthonormal basis for V,*

$$F(u) = \langle u, \sum_{i=1}^{n} \overline{F(v_i)} v_i \rangle.$$

2. Adjoints

Suppose both $(V, \langle\ ,\ \rangle_V)$ and $(W, \langle\ ,\ \rangle_W)$ are finite dimensional inner product spaces. If $A: V \to W$ is a linear map, and $w \in W$, the adjoint of A, A^*, is the map defined by requiring

$$\langle Av, w \rangle_W = \langle v, A^* w \rangle_V$$

for all $v \in V$.

Patrick's Concept Image

Suppose S is the set of items you want to buy. Then for each $s \in S$, $Q(s)$ is the quantity of s in the package of s and $P(s)$ is the price of s. Your final bill will be

$$\sum_{s \in S} Q(s) P(s).$$

The store owner wants to make you happy. He can either double $Q(s)$ for each s or half $P(s)$ for each s.

$\langle v_i, A^*(w) \rangle = \langle A(v_i).w \rangle$

$$A^*(w) = \sum_i \langle w, A(v_i) \rangle v_i$$

Moreover $(A^*)^* = A$ and $(AB)^* = B^*A^*$ if B is a linear map from an inner product space W into V.

Theorem. *"The" matrix of an adjoint of a linear map $L : V \to W$ is the conjugate transpose of "the" matrix for L.*

Proof.

3. Things to come

An operator $T\colon V \to V$ is called **self-adjoint** if

$$\langle Tv, u\rangle = \langle u, Tv\rangle.$$

Example. Suppose $V = \mathbf{R}^n$ with the standard inner product and L is an operator on V. Then L^*L is self-adjoint. Also L isself-adjoint if and only if its matrix w.r.t. the standard basis is symmetric ($M = M^{trsp}$).

Real Spectral Theorem. *Let L be a linear operator on a real inner product space V. Then V has an orthonormal basis of eigenvectors for L if and only if L is self-adjoint.*

So there are two orthonormal eigenvectors for $\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$.

Theorem. *If T is any operator there exists an isometry S and a self-adjoint operator A with positive eigenvalues such that*

$$A^2 = T^*T \text{ and } T = SA$$