

Linear Algebra

Lecture 17

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1. What is an inner product space?

An inner product space is a vector space with a notion of distance.

Namely, if $(V, \langle \cdot, \cdot \rangle)$ is an inner product space and $v, w \in V$ we say the distance between v and w , $d(v, w)$ is

$$\|v - w\|.$$

Then, we have the triangle inequality: If u, v and w are three vectors in V , then

$$d(v, u) \leq d(v, w) + d(w, u).$$

2. Orthonormal Bases

If $(V, \langle \cdot, \cdot \rangle)$ is a n -dimensional inner product space there is a basis v_1, \dots, v_n for V such that $\langle v_i, v_j \rangle = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$. Then, we saw, if $v = a_1v_1 + \dots + a_nv_n$ and $w = b_1v_1 + \dots + b_nv_n$,

$$\langle v, w \rangle = a_1\bar{b}_1 + \dots + a_n\bar{b}_n.$$

In particular, $v = \langle v, v_1 \rangle v_1 + \dots + \langle v, v_n \rangle v_n$.

Examples. (i) \mathbf{F}^2

(ii) $V = \{\cos(nx) : 0 \leq n \leq m\}$

$$\langle f, g \rangle = \frac{2}{\pi} \int_0^\pi f(x)g(x)dx.$$

3. Minimization Problems

Suppose you have a point v in space and you want to get as close to v as possible while staying on the ground. How close can you get and where is this place? To deal with this we need orthogonal projections.

Suppose $(V, \langle \cdot, \cdot \rangle)$ is an inner product space and U is a subspace. Let $U^\perp = \{v \in V : \langle u, v \rangle = 0\}$.

Theorem. U^\perp is a subspace of V and $V = U \oplus U^\perp$.

Proof.

If $v \in V$, $P_U(v)$ is defined to be the unique element of U such that $v - P_U(v) \in U^\perp$.

Proposition. Suppose U is a subspace of V and $v \in V$. Then, $P_U(v)$ is the closest element in U to v .

Lemma. P_U is linear.

Reading and problems for next time: Read pages 116-123. Do problems 17, 19 and 21 in §6.