Linear Algebra

Lecture 16

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1. Questions

If $T \in \mathcal{L}(V)$ is every eigenvector for T is in Null (T) or Range (T)?

How do we find eigenvalues?

What statements can we make about non-distinct eigenvalues?

What theorems can be extended to $\dim = \infty$?

2. Inner Products

Recall if W is a vector space over $\mathbf{F}$, an inner product $\langle\ ,\ \rangle$ on W is a map $\langle\ ,\ \rangle \colon W \times W \to \mathbf{F}$ which is linear in the first variable and "anti"-linear in the second, $\langle v, v\rangle$ is always real and ≥ 0 and is 0 only when both entries are 0. Moreover, $\langle v, w\rangle = \overline{\langle w, v\rangle}$.

A space with an inner product is called an **inner product space**. A good thing to observe is that every subspace of an inner product space is naturally an inner product space.

The **norm** of v, denoted $||v||$ is $\sqrt{\langle v, v\rangle}$. And as you showed for homework, if $V = \mathbf{R}^2$ and if θ is the angle between v and w,

$$\langle v, w\rangle = ||v||\,||w|| \cos\theta.$$

We will study operators which preserve lengths, **isometries**, or have **orthonormal** eigenbases.

Comments on Exercises

A. An operator N on a vector space is said to be nilpotent if $N^n = 0$. Show if V is finite dimensional and N is nilpotent, the matrix of N with respect to some basis of V is upper triangular with zeroes on the diagonal.

3.

$$\left(\sum a_j b_j\right)^2 \leq \sum (ja_j)^2 \sum \left(\frac{b_j}{j}\right)^2$$

5. Is $||(x_1, x_2)|| = |x_1| + |x_2|$ a norm?

7. Show

$$\langle u, v \rangle = \frac{||u+v||^2 - ||u-v||^2 + ||u+iv||^2 i - ||u-iv||^2 i}{4}.$$

3. Orthonormal Bases

We will show that if $(V, \langle \ , \ \rangle)$ is a n-dimensional inner product space there is a basis $(v_1, \ldots, v_n)$ for V such that

$$\langle v_i, v_j \rangle = \begin{cases} 1 & \text{if i=j} \\ 0 & \text{if } i \neq j \end{cases}.$$

Then, if $v = a_1 v_1 + \cdots + a_n v_n$ and $w = b_1 v_1 + \cdots + b_n v_n$,

$$\langle v, w \rangle = a_1 \bar{b}_1 + \cdots + a_n \bar{b}_n.$$

In particular, $v = \langle v, v_1 \rangle v_1 + \cdots + \langle v, v_n \rangle v_n$. Such a basis is called **orthonormal**.