Linear Algebra

Lecture 15

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1. Operators on complex and real vector spaces

There is an upper triangular matrix, with respect to some basis, for a linear operator L on a finite dimensional vector space V if and only if there exists a sequence of invariant subspaces $V_0 \subset V_1 \subset \cdots \subset V_m$ of V such that $V_0 = \{0\}$, $V_n = V$ and $\dim V_{k+1} - \dim V_k = 1$ if $0 < k+1 \leq n$.

Theorem. *Every linear operator L on a finite dimensional complex vector space V can be put in upper triangular form.*

Proof. Suppose λ is an e.v. and let $U = (L - \lambda)V$.

We saw that if L is a rotation of $\mathbf{R}^2$ about the origin of other that 0^o or 180^o, L has no eigenvectors, but this is the "worst" that can happen. That is,

Theorem. *Let L be a linear operator on a real vector space V. Then there exists a sequence of invariant subspaces $= \{0\} = V_0 \subset V_1 \subset \cdots \subset V_m = V$ of V such that $\dim V_{k+1} - \dim V_k$ is 1 or 2.*

If $\dim V = 4$ this theorem means we can find a basis such that the matrix of L looks like

$$\begin{pmatrix} a_1 & b_1 & * & * \\ c_1 & d_1 & * & * \\ 0 & 0 & a_2 & b_2 \\ 0 & 0 & c_2 & d_2 \end{pmatrix}$$

2. Inner Products

If W is a vector space over $\mathbf{F}$, an inner product $\langle\ ,\ \rangle$ on W is a map

$$\langle\ ,\ \rangle\colon W \times W \to \mathbf{F}$$

$$(v, w) \mapsto \langle v, w \rangle,$$

which is linear in the first variable and "anti-linear" in the second, $\langle v, v \rangle$ is always ≥ 0 and is 0 only when both entries are 0. Moreover,

$$\langle v, w \rangle = \overline{\langle w, v \rangle}.$$

Example. Suppose W is the space of polynomials over $\mathbf{F}$. Then set,

$$\langle p, q \rangle = \int_0^1 p(x)\overline{q(x)}dx.$$

Suppose $p(x) = a + bx$ and $q(x) = c + dx$. Then $\langle p, q \rangle =$

Of course, there is the dot product ("standard inner product") on $\mathbf{F}^n$

$$\langle a_1, \ldots, a_n, b_1, \ldots, b_n \rangle = a_1\bar{b}_1 + \cdots + a_n\bar{b}_n.$$

One defines the **norm** of v, denoted $||v||$ to be $\sqrt{\langle v, v \rangle}$. Eg., if $V = \mathbf{F}^n$ with standard inner product and $v = (a_1, \ldots, a_n)$,

$$||v|| = \sqrt{a_1^2 + \cdots + a_n^2}.$$

Reading and problems for next time: Read pages 106-111. Do problems 3, 5, 7 in §6.