Linear Algebra

Lecture 14

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1. Upper Triangular Matrices

Let L be the operator on $\mathbf{F}^2$ whose matrix (with respect to the standard basis) is

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

What are the eigenvectors?

Suppose L is a linear operator on V and $(v_1, \ldots, v_n)$ is a basis for V such that the matrix for L is

$$\begin{pmatrix} \lambda_1 & * & \cdots & * \\ 0 & \lambda_2 & \cdots & * \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

I.e. is **upper-triangular,** . Then, it follows that the k-dimensional subspace V_k spanned by $v_1, \ldots, v_k$ is invariant. In fact,

$$L(v_k) = \lambda_k v_k + w_k$$

for some $w_k \in V_{k-1}$.

Theorem. *All the λ_i are eigenvalues of L and L is invertible if and only if they are all non-zero.*

Proof. We prove the second part first. We proceed by induction on n.

Suppose $\lambda_1, \ldots, \lambda_{n-1}$ are non-zero. Suppose $\lambda_n = 0$.

Exercises. 4. $p(T)$ and $p'(T)$.

Show if $L \in \mathcal{L}(V, W)$ and $T \in \mathcal{L}(V, U)$ are surjective, and $\text{Null}\, L = \text{Null}\, T$ then U is isomorphic to W.

8. Find all eigenvectors of the backwards shift operator.

14. Show $p(STS^{-1}) = Sp(T)S^{-1}$.

15. Suppose $\mathbf{F} = \mathbf{C}$, $p(x)_{\neq 0} \in \mathbf{C}[x]$. Show a is an eigenvalue for $p(T)$ if and only if $a = p(\lambda)$ for some eigenvalue λ of T.

21. Suppose $P^2 = P$. Show $V = \text{null}P \oplus \text{range}P$.

Reading and problems for next time: Read pages 106-111. Do problems 1, 2, 4 in §6.