Linear Algebra

Lecture 12

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Suppose V is a vector space over $\mathbf{F}$ and L is an **operator** on V.

1. Eigenspaces

Theorem. *Suppose* $\lambda_1, \ldots, \lambda_n$ *are distinct elements of* $\mathbf{F}$ *and* $v_1, \ldots, v_n \in V$ *are* **non-zero** *eigenvectors such that* $Lv_i = \lambda_i v_i$*. Then, these vectors are independant.*

Proof. Suppose k is minimal such that $v_1, \ldots, v_k$ are dependant.

If $\lambda \in \mathbf{F}$, $\{v \in V : L(v) = \lambda v\}$ ls a subspace of V called an **eigenspace**.

2. Polynomials of Operators

If $P(X) = \sum_{i=1}^{n} a_i X^i \in \mathbf{F}[X]$,

$$P(L) = \sum_{i=1}^{n} a_i L^i \in \mathcal{L}(V, V).$$

Lemma. *If* V *is finite dimensional and* $v \in V$*, there exist a non-zero* $P \in \mathbf{F}[X]$ *such that* $P(L)v = 0$*.*

Proof.

Theorem. *If* $v_{\neq 0} \in V$, $P \in \mathbf{F}[X]$ *is non-zero such that* $P(L)v = 0$ *and is of minimal degree and* $P(\lambda) = 0$, *then* λ *is an eigenvalue of* L.

Proof.

3. Linear operators over C

Fundamental Theorem of Algebra. *Every polynomial of degree at least 1 over* **C** *has a root.*

Easy fact: Every polynomial over **R** of odd degree has a real root.

Theorem. *Every operator on a non-zero finite dimensional complex vector space* V *has an eigenvalue.*

Reading and problems for next time: Read pages 97-106. Do problems 20, 21 in §5.