Linear Algebra

Lecture 10

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1. Linear maps and Bases

Suppose $L\colon V \to W$ is a linear map, $(v_1, \ldots, v_n)$ is a basis for V and $(w_1, \ldots, w_m)$ is a basis for W. Then we can write

$$L(v_i) = \sum_k a_{i\,k} w_k.$$

Such data can be encoded by a matrix.

Example. Let V be the polynomials of degree at most 2 over $\mathbf{R}$, $W = \mathbf{R}^2$ and let

$$L(f) = (f(0), f'(0)).$$

Proposition. *Once bases are chosen, the matrix of a sum of linear maps is the sum of the matrices and the matrix of a product is the product of the matrices.*

2. Invertibility

A linear map $L\colon V \to W$ is said to be invertible if there exists a linear map $M\colon W \to V$ such that ML is the identity map of V and LM is the identity map of W.

Examples and non-examples. Suppose $V = \{f : f \text{ is continuously differentiable on } \mathbf{R}\}$ and $W = \{f : f \text{ is continuous on } \mathbf{R}\}$.

Proposition. *A linear map is invertible if and only if it is an isomorphism.*

Theorem. *If V is finite dimensional, a linear map $L\colon V \to V$ is invertible if and only if $\mathit{null}\, L = 0$.*

Proof.

Exercises: A. Suppose $L\colon V \to \mathbf{F}$ is a linear map and $v \notin \operatorname{null} L$. Show $V = \operatorname{null} L \oplus \operatorname{span}\{v\}$.

B. Suppose V is finite dimensional and $S \in \mathcal{L}(V, W)$ is surjective. Show there exists a $T \in \mathcal{L}(W, V)$ such that ST is the identity.

C. Suppose U is a subspace of V. Show, if V is finite dimensional, there exists a vector space W and linear map $L \in \mathcal{L}(V, W)$ such that $\operatorname{null} L = U$. Is this true if V is not finite dimensional?

16. Show $\dim \operatorname{null} ST \leq \dim \operatorname{null} S + \dim \operatorname{null} T$.

22. If V is finite dimensional. Show ST is invertible if and only if S is invertible and T is invertible.

26. Show there is only one solution of $\sum a_{ij} x_j = 0$ iff there is always a solution of $\sum a_{ij} x_j = c_i$.

For next time: Read pages 63-72. Do problems 4 and 5 in Chapter 4. Also; Show if $L \in \mathcal{L}(V, W)$ and $T \in \mathcal{L}(V, U)$ are surjective, and $\operatorname{null} L = \operatorname{null} T$ then U is isomorphic to T.