Final Problems

1. Suppose $A(t) = A + C(t)$ where A is a constant matrix whose eigenvalues have negative real parts, the entries of $C(t)$ are continuous on $[0, \infty)$ and $\lim_{t\to\infty} ||C(t)|| = 0$. Show the solutions of $X' = A(t)X$ are uniformly and asymptotically stable. (Hint: Let $\Phi(t)$ be a fundamental matrix for $X' = AX$. Verify there are positive constants α and R such that $||\Phi(t)|| < R\exp(-\alpha t)$ for $t \geq 0$. Suppose $R||C(t)|| \leq c < \alpha$ for $t \geq r$. Now use an argument similar to that used in Sanchez to show if $x(t)$ is a solution for $t \geq t_1$, $||x(t)|| \leq R\exp\big((c-\alpha)(t-t_1)\big)\exp\big(R\int_0^r ||C(s)||\mathrm{d}s\big)||x(t_1)||$.

Let

$$\mathcal{E}: \quad y'' + a(t)y = 0,$$

where $a(t)$ is continuous on $[0, \infty)$. Suppose

$$\int_1^\infty t|a(t)|\mathrm{d}t < \infty.$$

2. If $y(t)$ is a solution of $\mathcal{E}$ such that $\lim_{t\to\infty} y'(t) \neq 0$ and $w(t) = y(t)\int_t^\infty y^{-2}(s)\mathrm{d}s$, show $W(w, y) = 1$.

3. Show if $v(t)$ is a solution of $\mathcal{E}$ such that $\lim_{t\to\infty} v'(t) = 0$ then $\lim_{t\to\infty} tv'(t) = 0$.

4. Suppose A and B are constant 2×2 matrices such that the origin is a spiral point of (1) $X' = AX$ and a stable node of (2) $X' = BX$, Show there is no invertible matrix M with the property that if $x(t)$ is a solution of (1) $Mx(t)$ is a solution of (2).

5. True-False (if false give a counterexample)

(i) Suppose $A(t)$ is a continuous matrix on $[0, \infty)$ and that 0 is a stable solution of $X' = A(t)X$, then if M is an invertible constant matrix, 0 is a stable solution of $X' = MA(t)M^{-1}X$.

(ii) If $\mathcal{E}$ is an autonomous system on a domain D and $P \in D$ there are a infinite number of solutions with the whose trajectory passes through P.

(iii) If $\mathcal{E}$ is a linear differential equation on $\mathbf{R}^n$, 0 is a critical point.

(iv) If $\mathcal{E}$: $X' = MX$ where M is a constant 2×2 matrix and you know the eigenvalues of M, then you know the behavior of solutions of $\mathcal{E}$ near 0.

(v) Suppose A is a 2×2 symmetric matrix such that $Av \cdot v > 0$ for all non-zero vectors v. Then the origin is an unstable critical point of $X' = AX$.

(vi) Suppose M is a constant matrix and $\mathcal{E}$: $X' = MX + F(X)$ where $F(x)/||x|| \to 0$ as $||x|| \to 0$. Then if P is a proper node of $X' = MX$ it is a proper node of $\mathcal{E}$.

(vii) If $a(t)$ is continuous and $|a(t)| \leq t^{-3}$, for $t \geq 0$, then $x'' + a(t)x = 0$ has a bounded non-zero solution on $[0, \infty)$.

6. Show the autonomous plane system

$$u' = u - v - u^3 - uv^2, \quad v' = u + v - v^3 - u^2 v$$

has a unique critical point, which is unstable and a unique limit cycle.

7. Consider the system,

$$x' = ax + by, \quad y' = cx + dy \qquad a, b, c, d \in \mathbf{R}$$

$ad - bc = 0$. Show if one of a, b, c or d is not zero the phase plane of this system is a line of critical points and rectilinear trajectories either,

(a) approaching it or going away from it or (b) parallel to it.

8. Determine the nature and stability of the critical points of $x'' + (x')^3 + x = 0$.

9. Discuss the dependence on the sign of μ of the behavior near the critical point 0 of

$$u' = -v + \mu u^3, \quad v' = u + \mu v^3.$$

10. Suppose $A(t)$ is a square matrix of continuous functions on $[0, \infty)$ and $\int_0^\infty ||A(t)|| \mathrm{d}t < \infty$. Show every solution of the system $X' = A(t)X$ has a finite limit as $t \to \infty$.