Ordinary Differential Equations

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Lecture 9

Last time's homework

2.6 $x'' + a(t)x = 0$, $x_1(t)\int_{t_0}^{t} x_1(s)^{-2}\mathrm{d}s$

2.14 $x' = a(t)x + b(t)$, a and b periodic of period T. Let $h(t) = \exp(\int_{t_0}^{t} a(s)ds)$

$$f(t) = h(t)\big(x_0 + \int_{t_0}^{t} \frac{b(s)}{h(s)}\mathrm{d}s\big)$$

Suppose $t_0 = 0$ and $f(T) = f(0)$. Then,

$$\frac{\mathrm{d}}{\mathrm{d}t}f(t+T) = f'(t+T) = a(t+T)f(t+T) + b(t+T) = a(t)f(t+T) + b(t)$$

A. Suppose $x \in I$ and $L(t)$ and $M(t)$ are $n \times n$ matrices of continuous function on I such that $L'(t) = M(t)$, $L(x) = 0$ and $L(t)M(t) = M(t)L(t)$. (i) Show $F(t) = \exp(L(t)) := \sum_{n=0}^{\infty} \frac{L^n(t)}{n!}$ is a fundamental matrix for $X' = M(t)X$. (ii) Show $\det\exp(N) = \exp\ \mathrm{Trace}N$ for any $n \times n$ matrix.

Constant Coefficients

Suppose the entries of M are constant and consider

$$X' = MX. \tag{1}$$

If $\Phi(s)$ is a fundamental matrix for (1) $\Phi(0) = id$, $S^{-1}MS = \begin{pmatrix} \lambda_1 & & \dots & 0 \\ 0 & \lambda_1 & \dots & 0 \\ & & \ddots & \\ 0 & & \dots & \lambda_n \end{pmatrix}$,

$S\Phi(s)S^{-1} =$

$$\exp((S^{-1}MS)s) = \begin{pmatrix} \exp(\lambda_1 s) & \dots & & 0 \\ 0 & \exp(\lambda_2 s) & \dots & 0 \\ & & \ddots & \\ 0 & \dots & & \exp(\lambda_n s) \end{pmatrix} =: D$$

In particular, if M has a basis of eigenvectors $v_1, \ldots, v_n$ with eigenvalues $\lambda_1, \ldots, \lambda_n$ and S is the matrix

$$(v_1^T, \ldots, v_n^T),$$

$$\Phi(s) = S^{-1} D S.$$

But you can't diagonalize every matrix, even over the complex numbers. There is an invertible matrix S over $\mathbf{C}$ such that

$$S^{-1} M S = \begin{pmatrix} M_1 & \ldots & & 0 \\ 0 & M_2 & \ldots & 0 \\ \vdots & \ddots & & \\ 0 & \ldots & 0 & M_{r+2t} \end{pmatrix}$$

where

$$M_i = \begin{pmatrix} \lambda_i & 1 & 0 & \ldots & 0 \\ & \lambda_i & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda_i & 1 \\ & & & & \lambda_i \end{pmatrix}$$

and $\bar{\lambda}_i = \lambda_i$ if $i \leq r$ and $\overline{M_{r+2j}} = M_{r+2j-1}$ if $j > 0$. The λ_i are the eigenvalues of M over $\mathbf{C}$. It follows that

$$\Phi(s) = S \begin{pmatrix} \exp(M_1 s) & \ldots & & 0 \\ 0 & \exp(M_2 s) & \ldots & 0 \\ & & \ddots & \\ 0 & \ldots & \ldots & \exp(M_{r+2t} s) \end{pmatrix} S^{-1}.$$

Now,

$$\exp(M_i s) = \exp(\lambda_i s) \begin{pmatrix} 1 & s & \frac{s^2}{2} & \ldots & \frac{s^{n_i}}{n_i!} \\ & 1 & s & \ldots & \frac{s^{n_i - 1}}{(n_i - 1)!} \\ & & & \ddots & \\ & & & 1 & s \\ & & & & 1 \end{pmatrix}. \qquad (\textit{exercise})$$

Since, $\exp(a + bi) = \cos(a) + i \sin(b)$, if $\lambda_j = a_j + i b_j$ it follows that the entries of $\Phi(s)$ are linear combinations of

$$\exp(a_j s) \cos(b_j s) s^k \quad \text{and} \quad \exp(a_j s) \sin(b_j s) t^k,$$

where $k \leq n_j$. Thus if all the $a_i < 0$, the solutions tend to 0 as $s \to \infty$.

Homework for Next Time

Read pages 41-55b. If $X' = MX$ and L is a differentiable invertible matrix show $Y = LX$ satisfies $Y' = NY$ where $N = L'L^{-1} + LML^{-1}$. Suppose $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where a and b are differentiable and b doesdn't vanish. Show that if $X = \begin{pmatrix} y \\ z \end{pmatrix}$ is a solution,

$$\begin{pmatrix} y' \\ z' \end{pmatrix} = \begin{pmatrix} ay + bz \\ cy + dz \end{pmatrix}$$

y is twice differentiable and there exist u and v such

$$y'' = a'y + ay' + b'z + b(cy + dz)$$

$$z = b^{-1}(y' - ay)$$

$$y'' = uy' + vy.$$

$$\begin{pmatrix} y' \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix}$$

$$\begin{pmatrix} y' \\ y \end{pmatrix}' = \begin{pmatrix} u & v \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y' \\ y \end{pmatrix}$$

Deduce that there exists an L such that $L'L^{-1} + LML^{-1} = \begin{pmatrix} 0 & 1 \\ u & v \end{pmatrix}$.