Ordinary Differential Equations

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Lecture 7

Variation of Parameters

$$X'(t) = M(t)X(t) + B(t), \tag{1}$$

Suppose $\Phi(t)$ is a **fundamental matrix** for $X'(t) = M(t)X(t)$ (2) and $c(t)$ is an n-vector of functions. Then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\Phi(t)c(t)) &= \Phi'(t)c(t) + \Phi(t)c'(t)\\ &= M(t)\Phi(t)c(t) + \Phi(r)c'(t)\end{aligned}$$

so if

$$c'(t) = \Phi(t)^{-1}B(t)$$

$$\Phi(t)c(t)$$

solves (1). Thus,

Theorem. *Suppose $t_0 \in I$ and $\Phi(t)$ is a fundamental matrix for (2) such that $\Phi(t_0)$ is the identity matrix and $x_0 \in \mathbf{R}^n$. Then the solution x of (1) such that $x(t_0) = x_0$ is*

$$\Phi(t)x_0 + \Phi(t)\int_{t_0}^{t} \Phi(s)^{-1}B(s)\mathrm{d}s.$$

Corollary. *If $M(t)$ is a constant matrix, the solution Φ with $\Phi(0) = x_0$ is*

$$\Phi(t)x_0 + \int_0^t \Phi(t-s)B(s)\mathrm{d}s.$$

n-th Order Linear Equations

$$y^{(n)} + \cdots + a_0(t)y = 0 \tag{3}$$

We know how to transform this into a first order equation and after doing this we get from what we know about first order equations, assuming the EU-theorem, that the set of solutions is an n-dimensional vector space. A system of solutions $y_1, \ldots, y_n$ of (3) is said to be **fundamental** if and only if it is independent. The **Wronskian** $W(t)$ of $y_1, \ldots, y_n$ is

$$\det \begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & & & \vdots \\ y_1^{(n)} & y_2^{(n)} & \cdots & y_n^{(n)} \end{pmatrix}$$

$$y^{(n)} + \cdots + a_0(t)y = b(t) \tag{4}$$

Theorem. *Suppose $t_0 \in I$. Then the solution $y(t)$ of (4) such that*

$$y(t_0) = y^{(1)}(t_0) = \cdots = y^{(n-1)}(t_0) = 0$$

is

$$W(t_0)^{-1} \sum_{i=1}^{n} y_i(t) \int_{t_0}^{t} \Big(b(s) W_j(s) \exp\big(\int_{t_0}^{s} a_1(u) \mathrm{d}u \big) \Big) \mathrm{d}s$$

where W_j is the determinant of the matrix obtained from the Wronskian matrix by replacing it's j-th column by $(0, \ldots, 1)^T$.

Example. $y'' + y = f(t)$. A fundamental matrix is $\begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}$, $W(t) = 1$, $W_1(t) = -\sin(t)$ and $W_2(t) = \cos(t)$. Suppose $t_0 = 0 \in I$. Then our solution is,

$$-\cos(t) \int_0^t f(s) \sin(s) \mathrm{d}s + \sin(t) \int_0^t f(s) \cos(s) \mathrm{d}s.$$

Last time's homework

4. Make the substitution $y = y_1 \int u(s)\mathrm{d}s$.

8.

$$a_{i\,j} = \int_a^b u_i(t)u_j(t)\mathrm{d}t$$

Homework for Next Time

Read pages 33b-37. Do problems 2.6, 2.14 and

A. Suppose $x \in I$ and $L(t)$ and $M(t)$ are $n \times n$ matrices of continuous function on I such that $L'(t) = M(t)$, $L(x) = 0$ and $L(t)M(t) = M(t)L(t)$. (i) Show $F(t) = \exp(L(t)) := \sum_{n=0}^{\infty} \frac{L^n(t)}{n!}$ is a fundamental matrix for $X' = M(t)X$. (ii) Show $\det \exp(N) = \exp\ \mathrm{Trace} N$ for any $n \times n$ matrix.