Ordinary Differential Equations

Robert F. Coleman

Lecture 22

Uniform Stability

$$X' = A(t)X + F(t, X). \tag{1}$$

Theorem. *Suppose the solutions of $X' = A(t)X$ are uniformly stable and there exists a continuous function $b(t)$ on $[t_0, \infty)$ such that $||F(t, x)|| \leq b(t)||x||$ and*

$$\int_0^\infty b(s)\mathrm{d}s < \infty.$$

Then the solution $x(t) \equiv 0$ of (1) is uniformly stable.

Proof. $||\Phi(t)\Phi^{-1}(s)|| < M$. Suppose $x(t)$ is a solution of (1) for $t \geq t_1$. Then

$$x(t) = \Phi(t)\Phi^{-1}(t_1)x(t_1) + \int_{t_1}^t \Phi(t)\Phi^{-1}(s)F(s, x(s))\mathrm{d}s.$$

Now use Gronwall.

Examples. (i) Suppose A is a constant $n \times n$ matrix. Show the solutions of $X' = AX$ are uniformly stable if and only if 0 is a stable critical point of $X' = AX$.

$x'' - p(t)x' - q(t)x = f(t, x, x')$, $p(0) = 0$, $q(0) = -1$.

$$A(t) = \begin{pmatrix} 0 & 1 \\ q(t) & p(t) \end{pmatrix} \quad F(t, x, y) = \begin{pmatrix} 0 \\ f(t, x, y) \end{pmatrix}$$

If the solutions of $X' = A(t)X$ are also asymptotically stable then the solution $x(t) \equiv 0$ for (1) is as well.