Ordinary Differential Equations

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Lecture 20

Some non-linearity results

$$X' = A(t)X + F(t, X). \tag{1}$$

Suppose $A(t)$ is a continuous matrix on $[0, \infty)$ and F is continuous on $[0, \infty) \times B(a)$.

Theorem. *Suppose* $\lim_{x \to 0} ||F(t, x)||/|||x||| = 0$ *uniformly in* t*.If* $A(t) := A$ *is constant and stable then* $X(t) \equiv 0$ *is an asymptotically stable solution of (1).* *Proof. Let* $\Phi(t)$ *be a fund. matrix of* $X' = AX$*. Then* $\exists R, \beta > 0$ *s.t.* $||\Phi(t)|| \leq R \exp(-\beta t)$*.. Suppose* $f(t)$ *is a solution of* (1) *on* $[0, t_1)$ *and* $||f(0)|| < a$*. Then,*

$$f(t) = \Phi(t)f(0) + \int_0^t \Phi(t - s)F(s, f(s))\mathrm{d}s. \tag{2}$$

Suppose $0 < d < a$ *and* $||F(t, x)|| \leq m||x||$ *if* $||x|| < d$*,* $||f(t)||$ *is at most* d *on* $(0, t_1)$ *then*

$$||f(t)|| \leq R||f(0)|| \exp((mR - \beta)t).$$

If $m < \beta/R$ *and* $||f(0)|| < d/2R$ *we get*

$$||f(t)|| < d/2$$

for $0 \leq t < t_1$*. This and (2) implies* $\lim_{t \to t^-} f(t)$ *exists and is at most* $d/2$*.*

Now we need,

Theorem. *Suppose* $G(t, x)$ *is continuous and has continuous* x*-partials on some domain* B*. Suppose* $r(t)$ *is a solution of* $X' = G(t, X)$ *on* (a, b) *and* $y =: \lim_{t \to b^-} r(t)$ *exists and* $(b, y) \in B$*. Then the maximal interval for* r *contains* b*.*

Example.

$$x'' + x' + x = t(x^2 + (x')^2).$$

Uniform Stability

A solution $x(t)$ of (1) on $[t_0, \infty)$ is said to be **uniformly stable** if any solution sufficiently close to it at any time stay close to it for all time, i.e., for any $\epsilon > 0$ $\exists \delta > 0$, $\delta < a$, such that for any $t_1 \geq t_0$ and any local solution of (1) $x_1(t)$ at t_1 such that $|x_1(t_1) - x(t_1)|| < \delta$ extends to $[t_1, \infty)$ and satisfies

$$||x(t) - x_1(t)|| < \epsilon \text{ for } t \geq t_1.$$

A uniformly stable soultion on $[t_0, \infty)$ is stable at every $t \geq t_0$.

Example. $x' = a(t)x$. $x(t) \equiv 0$ is uniformly stable if and only if

$$\int_{t_1}^{t} a(s)\mathrm{d}s$$

is bounded above for all $t \geq t_1 \geq 0$.

We will prove,

Theorem. *Suppose the solutions of $X' = A(t)X$ are uniformly stable and there exists a continuous function $b(t)$ on $[0, \infty)$ such that $||F(t, x)|| \leq b(t)||x||$ and*

$$\int_0^\infty b(s)\mathrm{d}s < \infty.$$

Then the solution $x(t) \equiv 0$ of (1) is uniformly stable.

Example. On $[0, \infty] \times B(1)$,

$$x' = \frac{x + x^2}{(1+t)^2}.$$

Homework for Next Time

Read pages 102-104. Find a non-constant 2×2 matrix $A(t)$ and a non-zero vector of functions $F(t, x)$ satisfying the hypotheses of the last theorem and solve the equation.