Ordinary Differential Equations

Robert F. Coleman

Lecture 19

Comments on the proof of

Theorem. *Suppose* $t_0 = 0$. *If* $A(t) = A + C(t)$, *where* A *is a constant matrix whose eigenvalues have negative real parts and*

$$\int_0^\infty ||C(t)||\mathrm{d}t < \infty.$$

Then all solutions of $X' = A(t)X$ *are asymptotically stable.*

If $x(t)$ is a solution and $\Phi(t)$, $\Phi(0) = I$, is fundamental matrix of $X' = AX$.

$$x(t) = \Phi(t)x_0 + \int_0^t \Phi(t-s)C(s)x(s)\mathrm{d}s.$$

We know $||\Phi(s)|| \leq R\exp(-\beta t)$ because by lecture 9, if $\lambda_j = a_j + ib_j$ are the eigenvalues of A it follows that the entries of $\Phi(s)$ are linear combinations of $\exp(a_j s)\cos(b_j s)s^k \quad \text{and} \quad \exp(a_j s)\sin(b_j s)t^k.$

Gronwall's Lemma

Lemma. *Suppose* $u(t)$ *and* $v(t)$ *are non-negative and there is an* $\alpha \geq 0$ *such that*

$$u(t) \leq \alpha + \int_{t_0}^t v(s)u(s)\mathrm{d}s. \tag{1}$$

Then,

$$u(t) \leq \alpha \exp(\int_{t_0}^t v(s)\mathrm{d}s).$$

Proof. Suppose $\alpha > 0$. Then

$$\frac{u(t)v(t)}{\alpha + \int_{t_0}^t v(s)u(s)\mathrm{d}s} \leq v(t)$$

so

$$\log\big(\alpha + \int_{t_0}^t v(s)u(s)\mathrm{d}s\big) - \log\alpha \leq \int_{t_0}^t v(s)\mathrm{d}s.$$

Some non-linearity results

$$X' = A(t)X + F(t,X). \tag{2}$$

Suppose $A(t)$ is a continuous matrix on $[0,\infty)$ and F is continuous on $[0,\infty)\times B(a)$. Also suppose

$$\lim_{x\to 0} \frac{||F(t,x)||}{|||x|||} = 0 \tag{3}$$

uniformly in t. This implies $F(t,0)=0$.

Theorem. *If $A(t)$ is constant and stable then $X(t)\equiv 0$ is an asymptotically stable solution of (2).*

Proof. Suppose $x(t)$ is a local solution of (2) at $t=0$ and $x(0)=x_0$ (with $||x_0||<a$). Let $\Phi(t)$ be as above. Then,

$$x(t) = \Phi(t)x_0 + \int_0^t \Phi(t-s)F(s,x(s))\mathrm{d}s.$$

Suppose $||\Phi(t)|| \le R\exp(-\beta t)$. From (3), given $m>0$ there exists $d>0$ such that for $||x||<d$, $||F(t,x)|| \le m||x||$. There also exists a largest t_1 such that $x(t)$ is defined on $[0,t_1)$ and $||x(t)||<d$. Hence,

$$\exp(\beta t)||x(t)|| \le R||x_0|| + \int_0^t mR\exp(\beta s)||x(s)||\mathrm{d}s.$$

Now apply Gronwall's Lemma to get

$$||x(t)|| \le R||x_0||\exp((mR-\beta)t).$$

If $mR<\beta$ and $||x_0||<d/2R$ we get

$$||x(t)|| < d/2 \ \text{ for } t<t_1.$$

Now we need,

Theorem. *Suppose $G(t,x)$ is continuous and has continuous x-partials on some domain B. Suppose $r(t)$ is a solution of $X' = G(t,X)$ on (a,b) and $y =: \lim_{t\to b^-} r(t)$ exists and $(b,y)\in B$. Then the maximal interval for r contains b.*

Homework for Next Time

Read 99m-101. Show the R above is at least one (without using Gronwall's lemma).