Ordinary Differential Equations

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Lecture 18

Linear Systems

$$X' = A(t)X. \tag{1}$$

where $A(t)$ is continuous family of $n \times n$ matrices for $t \geq t_0$. Let $\Gamma = [t_0, \infty) \times \mathbf{R}^n$. Recall, if $g(t)$ is defined and differentiable and $(t, g(t)) \in \Gamma$ for $t \geq t_0$ and moreover suppose $g(t)$ is a solution of (1) for $t \geq t_0$. Then g is called **stable at** t_0 if,
(i) There exists $\gamma > 0$ such that if $h(t)$ is a solution of (1) near t_0, $||h(t_0) - g(t_0)|| < \gamma$, then $h(t)$ extends to a solution of (1) on $[t_0, \infty)$.
(ii) Given $\epsilon > 0$, $\exists \gamma \geq \delta > 0$ such that if h is as in (i) $||g(dt) - h(t)|| < eps$ for rcx$t \geq t_0$. The solution g is said to be **asymptotically stable at** t_0 if there exists $\gamma \geq \rho > 0$ such that when h is a solution of (1) near t_0 such that $||h(t_0) - g(t_0))|| \leq \rho$, then $\lim_{t\to\infty} ||g(t) - h(t)|| = 0$.

Theorem. *All solutions of (1) are stable if and only if all solutions are bounded.*

Suppose $\Phi'(t) = A(t)\Phi(t)$ and $\Phi(t_0) = I$.

Suppose all solutions are stable. Then the solution $x(t) = 0$ is stable so $\exists \delta > 0$ such that $||0 - \Phi(t)x|| < \epsilon$, if $||x|| < \delta$. Take $x = (0, \ldots, 0, \delta/2, 0, \ldots, 0)^T$. So

$$||\frac{\delta}{2}\Phi_i(t)|| < \epsilon.$$

we got

$$||\Phi(t)|| \leq 2n\epsilon/\delta.$$

Theorem. *Suppose $t_0 = 0$. If $A(t) = A + C(t)$, where A is a constant matrix whose eigenvalues have negative real parts and*

$$\int_0^\infty ||C(t)|| \mathrm{d}t < \infty.$$

Then all solutions of (1) are asymptotically stable.

We know this when $A(t) = A$.

Gronwall's Lemma. *Suppose $u(t)$ and $v(t)$ are non-negative and there is an $\alpha \geq 0$ such that*

$$u(t) \leq \alpha + \int_{t_0}^{t} v(s)u(s)\mathrm{d}s.$$

Then,

$$u(t) \leq \alpha \exp(\int_{t_0}^{t} v(s)\mathrm{d}s).$$

Example. $t_0 > 0$, $u(t) = t$, $v(t) = 1/t$, $\alpha = t_0$.

Proof of Theorem. Then if $x(t)$ is a solution of (1), $x(t)$ solves

$$x'(t) = Ax(t) + C(t)x(t)$$

so if $\Phi(t)$, $\Phi(0) = I$, is fundamental matrix of $X' = AX$.

$$x(t) = \Phi(t)x_0 + \int_0^t \Phi(t-s)C(s)x(s)\mathrm{d}s.$$

We know $||\Phi(s)|| \leq R\exp(-\beta t)$. Now apply Gronwall's lemma with

$$u(t) = ||x(t)||\exp(\beta t),\ v(t) = R||C(t)|| \text{ and } \alpha = R||x_0||.$$

1. $X' = A(t)X + B(t)$, $||x(t)|| \leq K$.

$||\Phi_i(t) + x(t)|| \leq J$, $||\Phi_i(t)|| = ||\Phi_i(t) + x(t) - x(t)|| \leq J + K$

$||\Phi(t)v|| < \epsilon$ if $||v|| < \delta$.

$$\Phi(t)v + x(t)$$

Homework for Next Time

Read 92t-97m. Do problems 3 and 6.