Ordinary Differential Equations

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Lecture 16

Approximation by Linear Systems

$$X' = F(X) \tag{1}$$

Suppose $(0,0)$ is a critical point of (1), the Jacobi matrix M of F at zero is invertible and $\lim_{||v||\to 0} \frac{F(v)-Mv}{||v||} = 0$.

Theorem. *The critical point* $(0,0)$ *is isolated,* $\lim_{||v||\to 0} \frac{||F(v)||}{||Mv||} = 1$ *and* $\lim_{||v||\to 0}(\arg F(v) - \arg Mv) = 0$.

Proof. Let $b = \inf_{||v||=1} ||Mv||$. $b \neq 0$. Then $||Mv|| \geq b||v||$, so

$$\lim_{||v||\to 0} \left|\left|\frac{F(v)}{||Mv||} - \frac{Mv}{||Mv||}\right|\right| = 0.$$

Examples. (i)

$$\theta'' + 2k\theta' + q\sin\theta = 0, \quad k > 0, q > 0.$$

This is (1) with $F(x,y) = \begin{pmatrix} y \\ -2ky - q\sin x \end{pmatrix}$. The critical points are $(n\pi, 0)$. Here at 0 and at π

$$M = \begin{pmatrix} 0 & 1 \\ -q & -2k \end{pmatrix} \quad M = \begin{pmatrix} 0 & 1 \\ q & -2k \end{pmatrix}.$$

(ii)

$$x' = y, \quad y' = 2x - x^2.$$

$$\begin{pmatrix} 0 & 1 \\ 2-2x & 0 \end{pmatrix}$$

Last time's homework

5. Sanchez says "given the system

$$x' = -x - \frac{y}{\log r},\ y' = -y + \frac{x}{\log r},$$

where $r = \sqrt{x^2 + y^2}$, show that $(0,0)$ is a simple critical point. Then use polar form to show that is a spiral point whereas it is a proper node for the corresponding linear system." You better first figure out what he means by $P(x,y) = -x - y\log^{-1} r$, and $Q(x,y) = -y + x\log^{-1} r$, make sure they are continuous and have continuous partials.

E.g., let $f(x,y) = \begin{cases} 0 & \text{(x,y)=(0,0)} \\ \frac{y}{\log r} & \text{otherwise} \end{cases}$. Then f is continuous and

$$\lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h} = \lim_{h\to 0} \frac{2}{\log h^2} = 0$$

while

$$\frac{\partial}{\partial y}\frac{y}{\log r} = \frac{1}{\log r} - \frac{y^2}{r^2 \log^2 r} \to 0 \quad \text{as } r \to 0.$$

Next compute the Jacobian matrix

$$\begin{pmatrix} \partial P(0,0)/\partial x & \partial P(0,0)/\partial y \\ \partial Q(0,0)/\partial x & \partial Q(0,0)/\partial y \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.,$$

so the associated linear equation has a proper node at 0. Now converting to polar coordinates,

$$r' = -r, \quad \theta' = \frac{1}{\log r}.$$

so

$$r(t) = \exp(-t + C) \ \text{ and } \theta(t) = -\log(t - C) + D,$$

and the 0 is a spiral point of the original equation.

Show for any $\alpha > 0$ $-1/(x^{1+\alpha}\log(x))$ is unbounded on $(0,1)$.

Homework for Next Time

Read the rest of Chapter 4. Do problem 6 and 8(f).