Ordinary Differential Equations

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Lecture 14

Linear Systems with Constant Coefficients

So $F(X) = MX$ where M is a constant 2×2 matrix. Suppose M is invertible. Then $(0,0)$ is the only critical point. When we worry about the behaviour of trajectories, we can assume $M =$

$$(a) \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad (b) \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \text{ or } (c) \begin{pmatrix} u & -v \\ v & u \end{pmatrix}.$$

In case (a), the trajectories are $\{(r\exp(\lambda_1 t), s\exp(\lambda_2 t))\}$.

If $\lambda_1\lambda_2 < 0$. The critical point is called a **saddle point**.

In case (b), the trajectories are $((ut+v)\exp(\lambda t), u\exp(\lambda t))$. The critical point is called a **node**, in this case. It is stable if $\lambda < 0$ and unstable if $\lambda > 0$.

Now suppose we are in case (c). (i) Suppose $u = 0$.

(ii) Suppose $u \neq 0$. Then the trajectories are $\{(c\exp(ut)\cos(vt+a), c\exp(ut)\sin(vt+a))\}$. The critical point is called a **spiral**. It is stable if $u < 0$ and unstable if $u > 0$.

Non-Linear Examples

Pendulum

$$\theta'' + 2k\theta' + q\sin\theta = 0$$

$$x' = y, \quad y' = -2ky - q\sin x$$

(b)

$$x' = y, \quad y' = 2x - x^2$$

(c)

$$x' = -x - y/\log(x^2 + y^2), \quad y' = -y + x/\log(x^2 + y^2)$$

Homework for Next Time

Do problem 4.