Ordinary Differential Equations

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Lecture 13

Some Homework

5. $t^n z^{(n)} + \cdots + a_n z = 0$ $t = e^u$.

6.

$$\int_a^b \Big(\int_a^x f(x,y)\mathrm{d}y\Big)\mathrm{d}x = \int_a^b \Big(\int_y^b f(x,y)\mathrm{d}x\Big)\mathrm{d}y$$

Let $g_n(u,v) = f(u)\frac{(v-u)^n}{n!}$ and $z_n(t) = \int_0^t g_{n-1}(u,t)\mathrm{d}u$.

Autonomous Systems

An n-dimensional system $X' = f(t,X)$ is said to be autonomous if $f(t,X) = F(X)$. $\blacksquare$
We suppose that the partials of F exist and are continuous on some region $\Gamma \subseteq \mathbf{R}^n$.

Basic Lemmas

$$X' = F(X) = \begin{pmatrix} P(X) \\ Q(X) \end{pmatrix} \tag{1}$$

Lemma. *Suppose $Y(t)$ is a solution of (1) at t_0. Then $Z(t) =: Y(t+c)$ is a solution of (1) at $t_0 - c$.*

Proof.

Lemma. *Through any point in Γ, there passes at most one trajectory.*

Proof. Suppose $Y(t_1) = Z(t_2)$.

If v is an isolated critical point of (1) then v is said to be **stable** if there exists a disk D around v such that every trajectory $T(t)$ of (1) which starts in B, is defined and stays in B for all time. If $\lim_{t\to\infty} T(t) = v$ for all trajectories, v is called **asymptocially stable**

Linear Systems with Constant Coefficients

So $F(X) = MX$ where M is a constant 2×2 matrix. Suppose M is invertible. Then $(0,0)$ is the only critical point.

http://www.math.uu.nl/people/beukers/phase/newphase.html

If L is an invertible constant matrix and Y is a solution of (1), we know LY solves

$$X' = (LML^{-1})X. \tag{2}$$

and vice versa, so if T is a trajectory of (2), $L^{-1}T$ is a trajectory of (1). So when we worry about the behaviour of trajectories, we can assume $M =$

$$(a)\ \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad (b)\ \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \text{ or } (c) \begin{pmatrix} u & -v \\ v & u \end{pmatrix}.$$

In case (a), the trajectories are $\{(r\exp(\lambda_1 t), s\exp(\lambda_2 t))\}$.

If $\lambda_1\lambda_2 < 0$. The cricial point is called a **saddle point**.

In case (b), the trajectories are $((ut+v)\exp(\lambda t), u\exp(\lambda t))$. The critical point is called a **node**, in this case. It is stable if $\lambda < 0$ and unstable if $\lambda > 0$.

Now suppose we are in case (c). (i) Suppose $u = 0$. (ii) $u \neq 0$. Then the trajectories are $\{(c\exp(ut)\cos(vt+a), c\exp(ut)\sin(vt+a))\}$. The critical point is called a **spiral**. It is stable if $u < 0$ and unstable if $u > 0$.

Homework for Next Time

Read 76b-81t. Do problems 1 and 3.