Ordinary Differential Equations

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Lecture 11

Some more comments about constant coefficients

If $f(x) = \sum_{i=1}^n b_i x^i$ is a polynomial and L is linear operator on a vector space V over $\mathbf{R}$ we let $f(L)$ be the operator

$$v \mapsto \sum_{i=1}^n b_i(\overbrace{L \circ \cdots \circ L}^{i \text{ times}})v.$$

Let D be the operator on the vector space of n-tuples of infinitely differentiable funtions,

$$(f_1, \ldots, f_n)^T \mapsto (f_1', \ldots, f_n')^T$$

then the equation $X' = MX$ can be rewritten

$$DX = MX. \tag{1}$$

If v is as solution of (1) so is Dv and

$$f(D)v = f(M)v.$$

Theorem. *Suppose $x_0 \in \mathbf{R}^n$ and $(M - \lambda I)^m x_0 = 0$, then if y is the solution of* (1) *such that $y(t_0) = x_0$ then $(D - \lambda I)^m y = 0$.*

Proof.

Corollary. *The entries of y are polynomials of degree at most m, times* $\exp(\lambda s)$.

Now lets put this into practice. Suppose

$$M = \begin{pmatrix} 2 & -1 & 3 \\ 1 & 0 & 2 \\ 0 & 0 & 2 \end{pmatrix}.$$

One eigenvector is $(0,0,1)^t$ with eigenvalue 2 which has multiplicity 1. This corresponds to the solution $(0,0,\exp(2s))^t$. Now try $(a,b,0)^t \exp(s)$.

Finally, try $(a_1 + a_2 s, b_1 + b_2 s, 0)^t \exp(s)$.

Last time's homework

1. $z^{(3)} + 7z^{(2)} + 16z' + 12z = 0$.

4. $y^{(2)} + q^2 y = A\sin(\omega t)$, $q \geq 0$.

$$-\sin(qt)\int^t \sin(\omega s)\cos(qs)\mathrm{d}s + \cos(qt)\int^t \sin(\omega s)\sin(qs)\mathrm{d}s$$

Autonomous Systems

An n-dimensional system $X' = f(t,X)$ is said to be **autonomous** if $f(t,X) = F(X)$. $\blacksquare$ We will suppose, as usual, that the partials of F exist and are continuous on some region $\Gamma \subseteq \mathbf{R}^n$. When $n = 2$ we can think of the equation $X' = F(X)$ as telling us that a partical subjected to the forces we impose at the point $(x,y) \in \Gamma$ must be moving at the velocity $F(x,y)$. The set Γ with arrows corresponding to these vectors is called the **phase space**.

Theorem. *Suppose $X(t)$ is a solution of* (1) *on a maximal interval I and there exist $t_1 \neq t_2$ such that $X(t_1) = X(t_2)$ then $I = \mathbf{R}$ and either X is constant or X is periodic.*

Homework for Next Time

Read 58b-65t. Do problems 5, 6 and 8.