Linear Algebra and Differential Equations

Robert F. Coleman

Final Problems

1. True/False If false, give a counterexample. If true just say true.

(i) Suppose M is an $n \times n$ matrix such that $\det(M) \neq 0$. Then the rows of M form a basis of $\mathbf{R}^n$.

(ii) The vectors in $\mathbf{R}^3$, $(1,1,1), (1,2,4), (1,3,9), (1,4,16)$ are independant.

(iii) There are infinitely many solutions (x, y) to the system of equations: $2x+y=3$, $-3x+y=-2$ and $4x+y=5$.

(iv) If M and P are $n \times n$ matrices and P is invertible then $(M) = (MP)$.

(v) If the characteristic polynomial of a square matrix has multiple roots, the matrix is not diagonalizable.

(vi) If A and B are square matrices such that AB is invertible, then both A and B are invertible.

(vii) Any set of orthonormal vectors in a finite dimensional inner product space can be completed to an orthonormal basis.

(viii) The dimension of the column space of an $m \times n$ matrix is at most m.

(ix) If A and B are commuting invertible square matrices and A is diagonalizable, so is B.

(x) Suppose p, q and g be continuous functions on an interval I and $f(x)$ and $h(x)$ are solutions of the equation $y'' + p(x)y' + q(x)y = g(x)$. Then $f(x) + h(x)$ is a solution.

2. Suppose M is a 3×3 matrix and the sum of the rows of M is zero. Show $\det(M) = 0$.

3. Find the displacement $u(x, t)$ of an elastic string of length π meters, fixed at both

ends, with no initial velocity, whose initial displacement is:

$$f(x) = \begin{cases} 0 & 0 \leq x \leq \pi/4 \\ x - \pi/4 & \pi/4 \leq x \leq \pi/2 \\ 3\pi/4 - x & \pi/2 \leq x \leq 3\pi/4 \\ 0 & 3\pi/4 \leq x \leq \pi \end{cases}$$

Assume the ratio a^2 of the tension of the string and the mass per unit length of the string is 1.

4. Suppose $v_1, \ldots, v_n$ and $w_1, \ldots, w_n$ are bases for a vector space V. Suppose $w_i = \sum_j a_{ij} v_j$ and $v_i = \sum_j b_{ij} w_j$. Let $A = (a_{ij})$ and $B = (b_{ij})$. Show $AB = I$.

5. (i) Complete $v_1 = (2, 0, 2), v_2 = (1, 2, 3)$ to a basis of $\mathbf{R}^3$. (ii) Determine the dimension of the span of $\{(1, 2, 3, 4), (1, 1, 1, 1), (1, 2, 3, 5), (3, 5, 7, 10)\}$.

6. (a) Compute the characteristic polynomial of

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a & b & c & 0 \end{pmatrix}.$$

(b) Write down a matrix whose characteristic polynomial is $T^4 + T + 1$. (c) Why is such a matrix always invertible?

7. Let A be a diagonalizable $n \times n$ matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$. Suppose that $|\lambda_1| < 1$ for every i. Prove that as $m \mapsto \infty$, all the entries of A^m approach zero.

8. Find a 2×2 constant matrix M over $\mathbf{R}$ such that

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-x}, \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{4x}$$

is a set of fundamental solutions of the equation $v' = Mv$.

9. Suppose A and B are commuting $n \times n$ matrices. (a) Show that if v is an eigenvector of B with eigenvalue b, so is Av. (b) Show that if B is diagonalizable and the diagonal entries are distinct then A is diagonalizable also.

10. Suppose a and l are positive real numbers. Show that the set of functions in two variables $u(x,t)$ on the region $0 \leq x \leq l$, $0 \leq t$ in the plane which are twice differentiable in x and in t and satisfy $u(0,t) = u(l,t) = 0$ and

$$a^2 u_{xx} = u_{tt}$$

is a vector space V. (b) Describe an infinite set of linearly independant elements of V and show the elements in this set are independant.

Final Problems Set 2

11. True/False If false, give a counterexample. (i) If the Wronskian of two differentiable functions (i.e. if a and b are the functions, their Wronskian is $a'b - b'a$) on I doesn't vanish at a point in I, it doesn't vanish at any point in I.

(ii) If y_1 and y_2 consitute a pair of fundamental solutions of $y'' + p(x)y' + q(x)y = 0$ on I. They can have no common zeroes.

(iii) The operator on differentiable functions $f \mapsto (f')^2$ is linear.

(iv) The function $e^{sin(x)}$ is periodic of period 2π.

(v) Suppose f is a continuously differentiable function on the interval $[0, \pi]$ which can be expanded in a series $a_0/2 + \sum_{n \geq 1} \big(a_n \cos(nx) + b_n \sin(nx)\big)$. Then $f = 0$ if and only if $a_n = b_n = 0$ for all n.

Suppose, in (vi)-(viii) below, that A and B are $n \times n$ matrices over $\mathbf{R}$.
(vi) If A and B commute and v is an eigenvector for both A and B, then v is an eigenvector for AB.

(vii) If A is invertible, then AB is invertible if and only if $\det(B) \neq 0$.

(viii) If A has an orthonormal eigenbasis, then A is symmetric.

(ix) Any set of three pairwise independant vectors in $\mathbf{R}^3$ is a basis for $\mathbf{R}^3$.

(x) If a real matrix M has no real eigenvalues, then the equation $v' = Mv$ has no real valued solutions.

12. Suppose M is a 2×2 matrix of continuous functions on I and N is an invertible 2×2 constant matrix. Suppose u is a solution of $v' = Mv$. (a) Show Nu is a solution of $v' = NMN^{-1}v$. (b) Show this is not always true if we don't suppose N is constant.

13. Find the solution of the equation

$$v' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} v$$

such that $v(0) = (1, 1, 1, 1)^T$.

14. Find the Fourier series of the function $f(x) = 1 + x^2$ on the interval $-1 \leq x \leq 1$.

15. Use the least squares method to find the orthogonal projection of $(1, 1, 1, 1, 0)$ on the solution space of $x_1 - x_2 + x_3 - x_4 = 0$ and $-x_1 - x_2 + x_3 + x_4 + x_5 = 0$.

17. True-False If false, give a counterexample.

(i) The set of common solutions of two linear homogeneous differential equations on an interval is a vector space.

(ii) Suppose V is an n dimensional vector space with an inner product. Then the maximal number of distinct pairwise orthogonal subspaces of V is n.

(iii) The determinant of an $n \times n$ $(n > 0)$ square matrix M is zero if and only if one of the rows of M is a linear combination of the other rows.

(iv) If M is a square matrix in row-eschalon form with a 1 in each row, then M is invertible.

(v) The product of two diagonalizable matrices is diagonalizable.

(vi) If v and w are two vectors in the plane and one knows $v \cdot w$, then one knows the angle between v and w.

(vii) If P is a plane and L is a line, both in $\mathbf{R}^3$, every vector in $\mathbf{R}^3$ is a sum of a vector in P and a vector in L.

(viii) Every vector space is spanned by some subset.

(vix) The function $x\log(x)$ does not satisfy a second order linear inhomogeneous equation.

(x) The functions on $\mathbf{R}$, $1, e^x$ and e^{2x}, are linearly independant.

18. Solve the equation

$$v'(T) = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} v(T) + \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

with the initial condition $v(0) = (0,0)^T$.

19. Let A be a 2×2 matrix. Suppose $A^n = 0$ for some integer n, show $A^2 = 0$.

20. Suppose A and B are two commuting diagonalizable matrices such that the eigenspaces for each are one dimensional. Show AB is diagonalizable.

21. Suppose V and W are spaces of continuous functions on $\mathbf{R}$ of dimensions r and s. Show the span of $\{vw : v \in V, w \in W\}$ is a vector space of dimension at most rs.

22. Let A be an $n \times n$ matrix. Define a linear map d_A from $\mathbf{M}_n$ to itself as follows:

$$d_A(X) = AX - XA.$$

(i) Show if A is symmetric the rank of d_A is at most $n^2 - n$. (ii) Show that if $cI \in Image(d_A)$, $c = 0$. (iii) Show $d_A(XY) = d_A(X)Y + Xd_A(Y)$.

23. Let A be a 2×2 matrix with distinct eigenvalues a and b. Show

$$e^A = (\frac{ae^b - be^a}{a-b})I + (\frac{e^a - e^b}{a-b})A.$$

(Hint: A can be written as BDB^{-1} where D is diagonal.)

24. Suppose V is a finite dimensional vector space. Let $V' = L(V, \mathbf{R})$ be the vector space of linear maps from V to $\mathbf{R}$. Let $\{e_1, \ldots, e_n\}$ be a basis for V. For $1 \le i \le n$ let e_i' be the element of V' such that $e_i'(e_j) = \delta_{ij}$ (Kronecker delta). (i) Show $\{e_1', \ldots, e_n'\}$ is a basis for V'. (ii) Suppose L is a linear map from V to itself, define $L' : V' \to V'$ by $L'(f)(v) = f(L(v))$ for all $v \in V$. Show L' is linear. (iii) If M is the matrix of L with respect to B, show A^T is the matrix for L' with respect to B'.

25. Solve the heat equations on $[0, 1]$;

$$(i) \qquad \begin{array}{c} u_t = u_{xx} \\ u(0,t) = 1 \quad u(1,t) = 0 \\ u(x,0) = x \end{array} \qquad \text{and} \qquad \text{(ii)} \quad \begin{array}{c} w_t = w_{xx} \\ w(0,t) = 0 \quad w(1,t) = 1 \\ w(x,0) = x. \end{array} .$$

What equation with what initial conditions does $au + bw$ satisfy where a and b are scalars.

26. Suppose f is a piecewise continuously differentiable function on the interval $[-1, 1]$ and w is a constant. Find a formal solution to the equation

$$y'' + w^2 y = f(x)$$

such that $y(0) = y'(0) = 0$ using the Fourier series of f.

27. If f has the Fourier series $\sum_{n=1}^{\infty} a_n \sin(nx)$ on the interval $[-\pi, \pi]$. Compute the integral

$$\int_{-\pi}^{\pi} (f(x))^2 dx$$

in terms of the a_n.

28. True or False. If false, give a counterexample.

(i) The sum of two projections is a projection.

(ii) Similar matrices have the same eigenvalues.

(iii) If A, B and C are 2×2 matrices, then $(AB - BA)^2$ commutes with C.

(iv) If V is an inner product space, W is a subspace of V and $L: V \to V$ is a linear map such that $L(W) \subseteq W$, then $L(W^{\perp}) \subseteq W$.

(v) Every isometry of $\mathbf{R}^n$ can be implemented by an orthogonal matrix.

(vi) For every linear map T from $\mathbf{R}^n$ to itself, there exists a positive integer k such that $ker(T^k) \cap Image(T^k) = \{0\}$.

(vii) Suppose A is an 2×2 matrix over $\mathbf{R}$ with no eigenvectors, then the equation $v' = Av$ has no solutions of the form $e^{at}w$, where a is a real number and w is a 2×1 column vector.

(viii) The Fourier series of a piecewise continuously differentiable function on the interval $[-1,1]$ always converges to the function.

(ix) Suppose f is the solution to an third order linear differentiable equation with continuous coefficients on the interval $[-1,1]$. Then if $f(0)=f'(0)=0$, f is identically zero.

29. Determine whether or not

$$\begin{pmatrix} 2 & -1 & 1 & 0 \\ 0 & 1 & 1 & 2 \\ -1 & 2 & 1 & 3 \\ 3 & 1 & 4 & 5 \end{pmatrix}$$

is invertible. If so compute the inverse, if not, compute the rank.

30. Suppose $B=\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$ and $C=\begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$ and

$$M = B\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}C.$$

Compute BC and use this to find the solutions $v(T)$ of the equation

$$v' = Mv$$

such that $v(0)=(1,0,0)^T$.

31. Solve the equation,

$$v'(t) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} v(t) + \begin{pmatrix} \cos(t) \\ -\sin(t) \end{pmatrix}$$

subject to the initial conditions $v(0)=(0,1)^T$.

32. Suppose $\mathbf{M}_n$ is the vector space of $n\times n$ matrices and $P\in\mathbf{M}_n$ is a projection (recall this means $P^2=P$). Show the vector subspace of $\mathbf{M}_n$, $P\mathbf{M}_nP$ has dimension 1 if and only if P has rank 1.

33. True-False If false give a counterexample

(i) If f is a continuously differentiable function on $[0,1]$ there is a unique function $u(x,t)$ such that $u_{x\,x} = u_t$ and $u(x,0) = f(x)$.

(ii) If p and q are continuous functions on $[0,1]$, the set of functions $f \in \mathcal{C}^2(I)$ such that $f'' + pf' + qf = 0$ is a vector space.

(iii) If M is a constant 2×2 matrix with no real eigenvalues, the solutions of $v' = Mv$ are bounded.

(iv) If f and g each satisfy a second order equation with constant coefficients so does fg.

(v) The set of quadratic forms on $\mathbf{R}^n$ is a vector space of dimension n^2.

34. Show if Q is a quadratic form on $\mathbf{R}^n$, $Q(v) = \langle v, v \rangle$ for an inner product $\langle\ ,\ \rangle$ if and only if the eigenvalues of the corresponding matrix are positive.