Linear Algebra and Differential Equations

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Lecture 38

II. Differential Equations

A. ODE's, $X' = F(t, X)$

1. Existence and uniquness of Solutions
2. Bases of Solutions of linear ODE's, $X' = A(t)X$

B. Systems of ODE's with Constant Coefficients

1. Distinct Roots
2. Repeated Roots
3. Complex roots

C. Fourier Series and PDE's

1. Piecewise Differentiable Functions

a. $\cos(\frac{n\pi}{L})$, $\sin(\frac{n\pi}{L})$.

b. Even and Odd Functions

c. $\langle f, g\rangle = \frac{2}{L}\int_0^L f(x)g(x)\mathrm{d}x$

D. PDE's with boundary value conditions

a. Heat Equation. $u_t = \alpha^2 u_{x\,x}$

i) $e^{-(\alpha\pi n/L)^2 t}$

ii) no flow

iii) zero^o ends

b. The Wave Equation, $u_{t\,t} = \alpha^2 u_{x\,x}$

E. Determing how many terms you need.

Final Problems

29. Determine whether or not

$$\begin{pmatrix} 2 & -1 & 1 & 0 \\ 0 & 1 & 1 & 2 \\ -1 & 2 & 1 & 3 \\ 3 & 1 & 4 & 5 \end{pmatrix}$$

is invertible. If so compute the inverse, if not, compute the rank.

30. Suppose $B = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$ and

$$M = B \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} C.$$

Compute BC and use this to find the solutions $v(T)$ of the equation

$$v' = Mv$$

such that $v(0) = (1, 0, 0)^T$.

31. Solve the equation,

$$v'(t) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} v(t) + \begin{pmatrix} \cos(t) \\ -\sin(t) \end{pmatrix}$$

subject to the initial conditions $v(0) = (0, 1)^T$.

32. Suppose $\mathbf{M}_n$ is the vector space of $n \times n$ matrices and $P \in \mathbf{M}_n$ is a projection (recall this means $P^2 = P$). Show the vector subspace of $\mathbf{M}_n$, $P\mathbf{M}_nP$ has dimension 1 if and only if P has rank 1.

33. True-False If false give a counterexample

(i) If f is a continuously differentiable function on $[0, 1]$ there is a unique function $u(x, t)$ such that $u_{xx} = u_t$ and $u(x, 0) = f(x)$.

(ii) If p and q are continuous functions on $[0, 1]$, the set of functions $f \in \mathcal{C}^2(I)$ such that $f'' + pf' + qf = 0$ is a vector space.

(iii) If M is a constant 2×2 matrix with no real eigenvalues, the solutions of $v' = Mv$ are bounded.

(iv) If f and g each satisfy a second order equation with constant coefficients so does fg.

(v) The set of quadratic forms on $\mathbf{R}^n$ is a vector space of dimension n^2.

34. Show if Q is a quadratic form on $\mathbf{R}^n$, $Q(v) = \langle v, v \rangle$ for an inner product $\langle\ ,\ \rangle$ if and only if the eigenvalues of the corresponding matrix are positive.