Linear Algebra and Differential Equations

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Lecture 10

Linear ODE's

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \qquad (*)$$

Definition. *If* $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ *is a solution of an* $(*)$, *then the set* $\{(x(t), y(t))\}$ *in the plane is its* **trajectory**.

If v and w are solutions of $(*)$, then so is $rv + sw$.

Suppose now a, b, c and d are constant. Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

If v is an eigenvector for M with eigenvalue λ, then $\exp(\lambda t)v$ is a solution of $(*)$.

Definition. *If* $(P - \lambda I)^n u = 0$ *for some positive integer* n, *then* u *is called a* **generalized eigenvector** *with eigenvalue* λ.

Proposition. *Suppose* $Mu = \lambda u + v$ *where* v *is an eigenvector for* M *with eigenvalue* λ, *then*

$$vte^{\lambda t} + ue^{\lambda t}$$

is a solution of $(*)$.

Proof.

Suppose S is invertible and w is a solution to $(*)$. Then Sw is a solution to

$$\mathbf{x}' = (SMS^{-1})\mathbf{x}.$$

This means we can assume M is in normal form.

Distinct real eigenvalues.

Suppose M has the eigenvalues $\lambda > \mu$ corresponding to the eigenvectors v and w. (Eg. $M = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$). Then the solutions of $(*)$ are $r\exp(\lambda t)v + s\exp(\mu t)w$, $r, s \in \mathbf{R}$.

Complex Conjugate Eigenvalues

Suppose $M = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, $b > 0$.

$$M \begin{pmatrix} 1 \\ \pm i \end{pmatrix} = (a \mp bi) \begin{pmatrix} 1 \\ \pm i \end{pmatrix}$$

$$\exp((a+bi)t) \begin{pmatrix} 1 \\ -i \end{pmatrix} + \exp((a-bi)t) \begin{pmatrix} 1 \\ i \end{pmatrix} = 2\exp(at) \begin{pmatrix} \cos(bt) \\ \sin(bt) \end{pmatrix}$$

$$-i(\exp((a+bi)t) \begin{pmatrix} 1 \\ -i \end{pmatrix} - \exp((a-bi)t) \begin{pmatrix} 1 \\ i \end{pmatrix}) = 2\exp(at) \begin{pmatrix} \sin(bt) \\ \cos(bt) \end{pmatrix}$$

One Real Eigenvalue

Scalars $M = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$ *or* $M = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$. *Then* $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ *is an eigenvector with eigenvalue* λ *, so one solution is* $\begin{pmatrix} \exp(\lambda t) \\ 0 \end{pmatrix}$.

$$M \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \lambda \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Let's look at

$$x_1' = 3x_1 - 2x_2$$

$$x_2' = 4x_2 - 6x_1$$

What is the solution with $(x_1(0), x_2(0)) = (1, 1)$*?*

Homework for Monday

Read 40b-45t. Do exercise 2.1.2(a).