

## Final Problems

1. Suppose  $A(t) = A + C(t)$  where  $A$  is a constant matrix whose eigenvalues have negative real parts, the entries of  $C(t)$  are continuous on  $[0, \infty)$  and  $\lim_{t \rightarrow \infty} \|C(t)\| = 0$ . Show the solutions of  $X' = A(t)X$  are uniformly and asymptotically stable. (Hint: Let  $\Phi(t)$  be a fundamental matrix for  $X' = AX$ . Verify there are positive constants  $\alpha$  and  $R$  such that  $\|\Phi(t)\| < R \exp(-\alpha t)$  for  $t \geq 0$ . Suppose  $R\|C(t)\| \leq c < \alpha$  for  $t \geq r$ . Now use an argument similar to that used in Sanchez to show if  $x(t)$  is a solution for  $t \geq t_1$ ,  $\|x(t)\| \leq R \exp((c - \alpha)(t - t_1)) \exp(R \int_0^r \|C(s)\| ds) \|x(t_1)\|$ .)

Let

$$\mathcal{E} : y'' + a(t)y = 0,$$

where  $a(t)$  is continuous on  $[0, \infty)$ . Suppose

$$\int_1^\infty t|a(t)| dt < \infty.$$

2. If  $y(t)$  is a solution of  $\mathcal{E}$  such that  $\lim_{t \rightarrow \infty} y'(t) \neq 0$  and  $w(t) = y(t) \int_t^\infty y^{-2}(s) ds$ , show  $W(w, y) = 1$ .

3. Show if  $v(t)$  is a solution of  $\mathcal{E}$  such that  $\lim_{t \rightarrow \infty} v'(t) = 0$  then  $\lim_{t \rightarrow \infty} tv'(t) = 0$ .

4. Suppose  $A$  and  $B$  are constant  $2 \times 2$  matrices such that the origin is a spiral point of (1)  $X' = AX$  and a stable node of (2)  $X' = BX$ , Show there is no invertible matrix  $M$  with the property that if  $x(t)$  is a solution of (1)  $Mx(t)$  is a solution of (2).

5. True-False (if false give a counterexample)

(i) Suppose  $A(t)$  is a continuous matrix on  $[0, \infty)$  and that 0 is a stable solution of  $X' = A(t)X$ , then if  $M$  is an invertible constant matrix, 0 is a stable solution of  $X' = MA(t)M^{-1}X$ .

(ii) If  $\mathcal{E}$  is an autonomous system on a domain  $D$  and  $P \in D$  there are a infinite number of solutions with the whose trajectory passes through  $P$ .

(iii) If  $\mathcal{E}$  is a linear differential equation on  $\mathbf{R}^n$ , 0 is a critical point.

(iv) If  $\mathcal{E}: X' = MX$  where  $M$  is a constant  $2 \times 2$  matrix and you know the eigenvalues of  $M$ , then you know the behavior of solutions of  $\mathcal{E}$  near 0.

(v) Suppose  $A$  is a  $2 \times 2$  symmetric matrix such that  $Av \cdot v > 0$  for all non-zero vectors  $v$ . Then the origin is an unstable critical point of  $X' = AX$ .

(vi) Suppose  $M$  is a constant matrix and  $\mathcal{E}: X' = MX + F(X)$  where  $F(x)/\|x\| \rightarrow 0$  as  $\|x\| \rightarrow 0$ . Then if  $P$  is a proper node of  $X' = MX$  it is a proper node of  $\mathcal{E}$ .

(vii) If  $a(t)$  is continuous and  $|a(t)| \leq t^{-3}$ , for  $t \geq 0$ , then  $x'' + a(t)x = 0$  has a bounded non-zero solution on  $[0, \infty)$ .

6. Show the autonomous plane system

$$u' = u - v - u^3 - uv^2, \quad v' = u + v - v^3 - u^2v$$

has a unique critical point, which is unstable and a unique limit cycle.

7. Consider the system,

$$x' = ax + by, \quad y' = cx + dy \quad a, b, c, d \in \mathbf{R}$$

$ad - bc = 0$ . Show if one of  $a, b, c$  or  $d$  is not zero the phase plane of this system is a line of critical points and rectilinear trajectories either,

(a) approaching it or going away from it or (b) parallel to it.

8. Determine the nature and stability of the critical points of  $x'' + (x')^3 + x = 0$ .

9. Discuss the dependence on the sign of  $\mu$  of the behavior near the critical point 0 of

$$u' = -v + \mu u^3, \quad v' = u + \mu v^3.$$

10. State and prove a version of Gronwall's lemma where  $u$  and  $v$  are only assumed defined and continuous on  $[t_0, t_1]$ .

11. Suppose  $A(t)$  is a square matrix of continuous functions on  $[0, \infty)$  and  $\int_0^\infty \|A(t)\| dt < \infty$ . Show every solution of the system  $X' = A(t)X$  has a finite limit as  $t \rightarrow \infty$ .

12. Use Liapunov's method to determine the type of stability or instability of  $x' = x - 3y + x^3$ ,  $y' = -x + y - y^2$ . (Hint: Consider  $V(x, y) = x^2 - 3xy - 3y^2$ .)

Can you do this by approximating by a linear equation? Explain your answer.

13. Suppose  $0$  is an isolated critical point and  $\sigma \equiv 0$  is a stable solution of  $\mathcal{E}$ . Show either  $\sigma$  is asymptotically stable or for every  $\epsilon > 0$  there is a full non-zero orbit contained in  $B(\epsilon)$ .

14. Determine whether the solution  $\sigma \equiv 0$  of the system  $x' = x(2y^3 - x^3)$ ,  $y' = -y(2x^3 - y^3)$  is asymptotically stable. ■

15. Show the singular points of the Legendre equation are regular and determine the first three terms of a power series solution at  $\infty$ .

16. Show the autonomous system  $x' = y$ ,  $y' = -\sin x - y$  has no cycles.

17. Suppose  $C$  is a cycle of an autonomous plane equation  $X' = F(X)$  where the components of  $F$  have continuous partials on  $\mathbf{R}^2$  and  $P$  is a point in the interior of  $C$ . (i) Show that the maximal interval of any solution  $\sigma$  such that  $\sigma(t_0) = P$  is  $\mathbf{R}$ .

18. Consider the system  $\mathcal{E}$ :  $x' = P(x, y)$ ,  $y' = Q(x, y)$ . Suppose  $P$ ,  $Q$  and  $F$  have continuous partials on a domain  $B$  and

$$dF(x, y) = H(x, y)(Q(x, y) - P(x, y)dy),$$

where  $H$  is a function with no zeroes on  $B$ . For  $c \in \mathbf{R}$ , let  $S_c = \{(x, y): F(x, y) = c\}$ . Show if  $S_c$  is contained in a closed, bounded subset of  $B$  and contains no critical points then it contains a cycle.

19. True-False. If false give a counterexample.

(i) Every critical point of an autonomous equation is either stable or unstable.

(ii) Suppose  $A(t)$  is continuous for  $t \geq 0$ . If the solutions of  $X' = A(t)X$  are uniformly stable then  $\sigma \equiv 0$  is uniformly stable for

$$X' = A(t)X + \exp(-t)X.$$

(iii) Suppose  $a(t)$  is continuous for  $t > 0$  and  $\lim_{t \rightarrow \infty} a(t) = 0$ . Then if  $y(t)$  is a solution of  $x'' + a(t)x = 0$ ,  $\lim_{t \rightarrow \infty} y'(t)$  exists.

(iv) If  $A$  and  $B$  are commuting constant  $n \times n$  matrices such that the solutions of  $X' = AX$  and  $X' = BX$  are asymptotically stable, then the solutions of  $X' = (A + B)X$  are asymptotically stable.

(v) A normalized second order equation with coefficients analytic at 0 has a basis solutions analytic at 0.

(vi) If  $\mathcal{E}$  is a differential equation with domain  $\mathbf{R}^n$  any solution  $\sigma$  of  $\mathcal{E}$  either has  $\mathbf{R}$  as its maximal interval of existence or  $\|\sigma\|$  is unbounded on its maximal interval.

(vii) Suppose  $\mathcal{E}: X' = F(X)$  is an autonomous plane equation. and the origin is a critical point. If there exists two distinct solutions  $\sigma, \sigma \neq 0$ , such that  $\lim_{t \rightarrow \infty} \sigma(t) = 0$ , then  $\sigma \equiv 0$  is stable.

(viii) If  $A$  is a  $2 \times 2$  matrix and 0 is a saddle point,  $\sigma$  is a solution  $\sigma \neq 0$ ,  $\lim_{t \rightarrow \infty} \sigma(t) \neq 0$ .

(xi) If  $\mathcal{E}$  has domain  $\mathbf{R}^2$ , then any solution has maximal interval of existence  $\mathbf{R}$ .

(x)  $\sigma(t)$  is a uniformly stable solution of the equation  $X' = F(t, X)$  if and only if  $\tau \equiv 0$  is a uniformly stable solution of  $X' = F(t, X + \sigma(t)) - F(t, \sigma(t))$

Suppose

$$\mathcal{E}: X' = F(X)$$

is a 2-dimensional autonomous system on  $\mathbf{R}^2$  and  $C$  is a periodic orbit of  $\mathcal{E}$ . Let  $B$  be the interior of  $C$

20. Suppose  $x(t)$  is a solution of  $\mathcal{E}$  such that  $x(0) \in B$ . (a) Show the maximal interval of  $x(t)$  is  $\mathbf{R}$ . (b) Show if  $D$  is the full orbit of  $x(t)$ , one of  $L(D^+)$  or  $L(D^-)$  is a periodic orbit distinct from  $C$  or contains a critical point of  $\mathcal{E}$  by considering the equation  $X' = -F(X)$ .