Ordinary Differential Equations

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Lecture 9

old homework

Suppose a and b are periodic of period T. Let $h(t) = \exp(\int_{t_0}^{t} a(s)ds)$. Show $x' = a(t)x + b(t)$ has a **unique** periodic solution iff $h(T) \neq 1$.

$$f(t) = h(t)\big(x_0 + \int_{t_0}^{t} \frac{b(s)}{h(s)}\mathrm{d}s\big)$$

Suppose $t_0 = 0$. If $f(T) = f(0)$, and $h(T) \neq 1$

$$x_0 = \frac{1}{1 - h(T)} \int_0^T \frac{b(s)}{h(s)}\mathrm{d}s$$

$$\frac{\mathrm{d}}{\mathrm{d}t} f(t+T) = f'(t+T) = a(t+T)f(t+T) + b(t+T) = a(t)f(t+T) + b(t)$$

and $f(0+T) = f(0)$.

If $h(T) = 1$ and $\int_0^T \frac{b(s)}{h(s)}\mathrm{d}s = 0$

$a(t) = \sin t$, $b(t) = 0$, $T = 2\pi$

Review of Last Time

Suppose M is a constant matrix with eigenvalues $\lambda_1, \dots, \lambda_m$ where $\lambda_j = a_j + ib_j$. Then if Y(s) solves

$$X' = MX, \tag{1}$$

the entries of $Y(s)$ are linear combinations of $\exp(a_j s)\cos(b_j s)s^k$ and $\exp(a_j s)\sin(b_j s)s^k$, where k is less than the multiplicity of λ_j.

Theorem. *(i) If $a_j < 0$ for all j, $\lim_{s\to\infty} Y(s) = 0$. (ii) If $a_j \leq 0$ for all j and $a_j < 0$ for all j such that the multiplicity of λ_j is greater than 1, then $Y(s)$ is bounded as $s \to \infty$.*

Proposition. *If (1) corresponds to the n-th order equation $y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = 0$, the characteristic polynomial of M is $x^n + a_1 x^{n-1} + \cdots + a_n$.*

Example. $y^{(3)} + 2y^{(2)} + 2y^{(1)} = 0$.

Some useful facts about polynomials.

Let $f(x)$ be a polynomial over $\mathbf{R}$.

Proposition. *Suppose there exists polynomials $h(x)$ and $g(x)$ such that $g(x)f(x) + h(x)f'(x) = 1$, then f has no multiple roots.*

Proof.

Example. $f(x) = x^3 + x + 1$.

Fundamental Theorem of Algebra. *There exist polynomials over* $\mathbf{R}$, $g_1, \dots, g_r$ *and $h_1, \dots, h_s$, such the g_j have degree one, the h_j have degree two and*

$$f(x) = g_1(x) \cdots g_r(x) h_1(x) \cdots h_s(x).$$

Proposition. *If the real parts of the roots of f are negative, then all the coefficients of f are positive.*

Proof.

Proposition. *Suppose $f(x) = x^3 + a_1x^2 + a_2x + a_3$. Then the real parts of the roots of f are negative if and only if $a_j > 0$ and*

$$a_1a_2 - a_3 > 0.$$

Example. The solutions of $y^{(3)} + 2y^{(2)} + y^{(1)} + y = 0$ tend to 0.

Homework for Next Time

Read 55b-58t. (A) Let M and N be $n \times n$ matrices of continuous functions on an interval I. Show that there exists an invertible $n \times n$ matrix L of differentiable functions on I such that if $Y' = MY$, then $(LY)' = N(LY)$ (Hint: First do this When $M = 0$.). Can one requiure the entries of L to be bounded? Why or why not? Do problems 1 and 4 in Chapter 3.s