Ordinary Differential Equations

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Lecture 5

Fundamental Solutions

$$X'(t) = M(t)X(t) + B(t), \qquad (*)$$

Suppose $B = 0$. Let $P \in I$. The set of solutions V_P of $(*)$ in a neighborthood of $(*)$ forms a vector space.

Theorem. *The map $e_P\colon V_P \to \mathbf{R}^n$, $e_P\colon X(t) \mapsto X(P)$, is linear and bijective.*

For $P \in I$, let J_P be the intersection of the maximal intervals of the elements of V_P.

Corollary. *J_P is an interval.*

Proof.

Corollary. *For any $Q \in J_P$, e_Q is an isomorphism.*

Proof.

Theorem. *The maximal interval for any solution of $(*)$ (with $B = 0$) is I.*

Proof. Suppose $J_P = (a, b)$ and $a \in I$. Suppose $Q \in J_P \cap J_a$ and $X \in V_P$.

A basis of solutions of $(*)$ is called a set of **fundamental solutions**.

Uniqueness

We will deal with the $n = 2$ case and write our equation as

$$u'' + p(x)u' + q(x)u = 0. \qquad (**)$$

We will need,

Lemma. *Suppose f is a positive differentiable function on an open interval I, $P \in I$ and $f'(x) \leq Kf(x)$. Then,*

$$f(x) \leq \exp(K(x-P))f(P) \quad \text{for } x \geq P.$$

Theorem. *If $P \in I$ and $(r,s) \in \mathbf{R}^2$ equation $(**)$ has at most one solution u satisfying $u(P) = r$ and $u'(P) = s$.*

Proof. Suppose $(r,s) = (0,0)$. Let $\sigma(x) = u^2 + (u')^2$. Then, $|2uu'| \leq \sigma$

$$\sigma'(x) = -2p(x)(u')^2 + 2(1-q(x))uu',$$

so

$$\sigma' \leq 2|p|(u')^2 + |1-q|\sigma \leq (1 + 2|p| + |q|)\sigma.$$

Proof of Lemma

Homework for Friday

Finish proof of the above lemma. Prove uniqueness for second order non-homogeneous linear ODE's. do problems 1 and 2 of chapter 2.

Read pages 21m-25b