Ordinary Differential Equations

Robert F. Coleman

Lecture 35

Poincaré-Bendixson

Suppose

$$\mathcal{E}: \qquad X' = F(X)$$

is a two dimensional autonomous system where F is defined on a bounded open subset D of the plane and its partials are continuous.

Suppose C^+ is a **positive semiorbit** of $\mathcal{E}$, i.e., $\exists t_0 \in \mathbf{R}$ and a solution $\phi(t)$ of $\mathcal{E}$ defined for $t \geq t_0$ such that

$$C^+ = \{\phi(t) \colon t \geq t_0\}.$$

A point $Q \in \mathbf{R}^2$ is said to be a **limit point** of C^+ if there exists a sequence of real numbers $\{t_n\}$ such that $t_n \to \infty$ and

$$\phi(t_n) \to Q.$$

The set of limit points of C^+ is called a **limit set** and is denoted $L(C^+)$. Negative semiorbits are defined similarly and also have limit points.

A full orbit C is the union of a negative semi-orbit and a positive semi-orbit and thus has a negative limit set $L^-(C)$ and a positive limit set $L^+(C)$. Set $L(C) = L^+(C) \cup L^-(C)$.

Poincaré-Bendixson Theorem. *If C^+ is contained in a closed subset of D and $L(C^+)$ consists of regular points only then $L(C^+)$ is a periodic orbit.*

Proposition. *Suppose 0 is an isolated critical point and $\sigma \equiv 0$ is a stable solution of $\mathcal{E}$. Then either σ is asymptotically stable or for every $\epsilon > 0$ there is a full non-zero orbit contained in $B(\epsilon)$.*

Limit Sets

Suppose C^+ is a positive semi-orbit contained in in a closed subset K of D. Suppose $C^+ = \{\phi(t)\colon t \geq t_0\}$ for a solution ϕ.

Theorem. *The limit set $L =: L(C^+)$ is a non-empty, closed, connected set.*

Proof. Compactness implies $L \neq \emptyset$.

Suppose $Q = \lim_{n\to\infty} Q_n$, $Q_n \in L$. Then, $Q_n = \lim_{m\to\infty} \phi(t_{n_m})$ where $t_{n_m} \geq t_0$ and $\lim_{m\to\infty} t_{n_m} = \infty$.

Now suppose $L = M \cup N$, $M \cap N = \emptyset$, $M \neq \emptyset$ and $N \neq \emptyset$, M and N both closed (and hence compact). Let $\delta = \text{dist}(M, N)$. Let $F(t) = \text{dist}(\phi(t), M)$. Claim: There exists $t_n \to \infty$, $F(t_n) = \delta/2$.

Lemma. *Suppose σ_n are solutions of $\mathcal{E}$ and $\{\sigma_n(0)\}$ converges to a regular point P. Then if σ solves $\mathcal{E}$, $\sigma(0) = P$, $\sigma_n(t) \to \sigma(t)$.*

Theorem. *Suppose L contains a regular point Q. Then, L contains a (and hence is the) full orbit through Q.*

Proof. Let σ be the the solution of $\mathcal{E}$ such that $\sigma(0) = Q$ defined on a maximal interval I.

Let $Q = \lim \phi(t_n)$, $t_n \to \infty$. Let $\sigma_n(t) = \phi(t + t_n)$. It follows that

$$\sigma_n(t) \to \sigma(t) \quad \text{for } t \in I.$$

(Note: $\sigma_n(t)$ is defined for $t \geq t_0 - t_n$.)

Such an orbit is called a **limit orbit**.

Homework for Next Time

Read pages 391b-394b. Do problems 1 and 2. (I think the hint should be: Show $\phi_1^2 + \phi_2^2 \to 1$.) Prove the above lemma.