Ordinary Differential Equations

Robert F. Coleman

Lecture 30

Clarification

$$\begin{aligned} x_0 + \int_{t_0}^{t} F(s,u(s))\mathrm{d}s &= x_0 + \int_{t_0}^{b} F(s,u(s))\mathrm{d}s + \int_{b}^{t_0} F(s,u(s))\mathrm{d}s \\ &= w + \int_{b}^{t} F(s,u(s))\mathrm{d}s \end{aligned}$$

Maximal Intervals

Theorem. *Suppose* $(t_0, x_0) \in B$ *and* $M(t_0, x_0) = (u, v)$, $v < \infty$, x *is the solution of* $\mathcal{E}$ *on* $M(t_0, x_0)$ *such that* $\mathrm{x}(t_0) = x_0$ *and* U *is a closed bounded subset of* B, *then* $\exists r > 0$ *such that* $\mathrm{x}(t) \notin U$ *if* $|v - t| < r$.

Proof. There exists $p > 0$ such that $||x - y|| \geq p$ for $x \in U$ and $y \in V = \mathbf{R}^n - B$.

If $(t_1, x_1) \in U$, $||(t, x) - (t_1, x_1)|| < p$ implies $(t, x) \in B$. There exist a finite number of points y_i in U such that

$$U \subset \bigcup_i B(y_i, p/2).$$

Let

$$U^* = \bigcup_i B[y_i, p/2] \quad \text{then } U \subset U^* \subset B.$$

There exist $m > 0$ and $k > 0$ such that

$$||F(t,x)|| \leq m \quad \text{and} \quad ||F(t,x) - F(t,y)|| \leq nk||x - y||$$

for $x, y \in U^*$. Choose $a + b < p/2$. Suppose r is such that

$$(i)\ r \leq a, \quad (ii)\ r \leq b/m \quad \text{and} \quad (iii)\ r \leq 1/nk.$$

Then for any point $(t_1, x_1) \in U$, $I[t_1, r] \subset M(t_1, x_1)$.

Let $v > t_1 > v - r$. Since $t_1 + r > v$, $(t_1, x(t_1)) \notin U$.

Example. Suppose $F(t, x)$ is a polynomial in t and x. Then solutions which don't extend to the whole real line are unbounded. Eg. $F(t, x) = 1 + x^2$.

What Every Gentleperson should know about compactness

Suppose T is a topological space.

Then T is said to be compact if whenever $T = \bigcup_{U \in \mathbf{C}} U$ where $\mathcal{C}$ is a collection of open subsets of T there exist a finite subset $U_1, \ldots, U_n$ of $\mathcal{C}$ such that

$$T = U_1 \cup \cdots \cup U_n.$$

Examples.

The compact subsets of $\mathbf{R}^n$ are the closed and bounded subsets. If $f\colon X \to Y$ is a continuous map, if T is a compact subset of X, $f(T)$ is a compact subset of Y.

This implies that continuous functions on closed intervals are bounded and achieve their extrema.

Lemma. *If $\mathcal{C}$ is a collection of closed set in a compact space K such that any finite subcollection of $\mathcal{C}$ has a non-empty intersection, then $\bigcap_{X \in \mathcal{C}} X \neq \emptyset$.*

Proof. $(\bigcap_{X \in \mathcal{C}} X)^c = \bigcup_{X \in \mathcal{C}} X^c$

Theorem. *Every sequence of points in a compact space has a convergent subsequence.*

Homework for Next Time

Read 131b-136m. Do problems 4 and 5.