Ordinary Differential Equations

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Lecture 2

Normal First Order DE's

These are

$$y' = F(x, y) \qquad (*)$$

Eg. $y' = -x/y$.

Fundmental Theorem of Calculus. *If $g(x)$ is continous on the interval $[a, b]$ and $c \in \mathbf{R}$ there is one and only one solution $y_c(x)$ to the equation*

$$y' = g(x)$$

such that $y_c(a) = c$. It is

$$c + \int_a^x g(t)\mathrm{d}t$$

Note that $y_{c+d} = d + y_c$.

Suppose $f(x)$ is a solution to $y' = g(y)$ such that $f_c(a) = c$. Then $h(x) = f(x+d)$ is also a solution and f and $h(a) = f(a + d)$.

Some general remarks

If x is a function in one variable into $\mathbf{R}^n$,

$$x(t) = (x_1(t), \ldots, x_n(t)),$$

we say x is differentiable if and only if all the x_i are and set

$$x'(t) = (x_1'(t), \ldots, x_n'(t)).$$

A k-th order equation in "normal form" is

$$x^{(k)} = G(t, x, x', \ldots, x^{(k-1)})$$

where G is a function of $1+nk$ variables into $\mathbf{R}^n$. E.g.,

$$x^{(2)} = \exp(tx + (t')^2) = G(t, x, x')$$

where $G(a, b, c) = \exp(ab + c^2)$. This is the same as the system

$$x_2'(t) = x_1(t)$$
$$x_1'(t) = G(t, x_2(t), x_1(t))$$

so that if $y = (x_1, x_2)$

$$y'(t) = (H_1(t, y), H_2(t, y))$$

where $H_1(a, (b_1, b_2))) = G(a, b_2, b_1)$ and $H_2(a, (b_1, b_2))) = b_1$.

In this way we only need to consider first order equations.

$$y' = f(t, y) \qquad (*)$$

If $\phi = (\phi_1, \ldots, \phi_n)$ is a solution to $(*)$ on an interval $I =: (a, b)$ the set $\{(t, \phi(t)) : t \in I\}$ is called an integral curve to $(*)$.

Last time's homework

$$F(x) = \int_{x_0}^{x} \frac{1}{f(s)} \mathrm{d}s$$

$f(s) \neq 0$. Suppose $x(t) = F^{-1}(t - c)$.

Existence and Uniqueness

Theorem. *Suppose $F(t, x)$ is defined on some domain $B \subseteq \mathbf{R}^{n+1}$. Suppose the partials of f, $\partial f/\partial x_i$ are defined and continuous on B, $i = 1, \ldots, n$. Then for every point $(u, v) \in B$ there exists a unique solution $\phi(t)$ of $(*)$ satisfying $\phi(u) = v$ and defined in some neighborhood of u.*

Homework for Friday

Read pages 4m-8t of Sanchez. Do problems 1, 3 and 5 of Chapter 1.