Banach Algebras and Spectral Theory

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The following are notes to a course taught in 1996. They have not been seriously edited, some things are repeated because they weren't proven n the lecture where they first appeared and the reader will see that attempts were made to correct mistakes made at one point (especially in the homework assignments) at a later point. Lectures 1-26 are based on notes of Arveson although I take credit for any errors. The remaining lectures are derived from Gohberg-Krejn, Introducuction á la theorie des Opérateurs linéaires non auto-adjoints dans un espace Hilbertien; Grothendieck, La Théorie de Fredholm; Riesz-Nagy, Functional Analysis; Conway, A Course on Functional Analysis and Dunford and Scwartz, Linear Operators, Part II.

Lecture 1

The prerequisites for this course include linear algebra and some familiarity with real and complex analysis as, well as measure theory. You should have seen Banach and Hilbert spaces before. Some theorems that will be used without proof are:

The Baire category theorem   The Hahn-Banach Theorem

The open mapping theorem   The closed graph theorem

The Stone-Weierstrass Theorem and The Riesz-Marcov Theorem

A good reference for some of this material is Linear Operators: Part I by Nelson Dunford and Jacob Schwartz as well as Notes on Measure Theory and Integration on Locally Compact Spaces by William Arveson.

There will be weekly to biweekly homework assignments. My office hours will be Monday 4-5 and Wednesday 3-4. My email address is coleman@math.

Suppose V is a Banach space. This course will concern itself with continuous linear operators $L : V \to V$. The basic problems are: given $v \in V$, determine whether the equation

$$Lw = v$$

has a solution, find one or get as close as you can to doing so.

The finite dimensional situation is one of the main motivations for the development of the theory.

In the finite dimensional case you are given an $n \times n$ matrix $M = (a_{ij})$ and an n-vector $g = (g_1, \ldots, g_n)$ and you want to solve the equation $Mf = g$, i.e., find $(f_1, \ldots, f_n)$ such that

$$\sum_j a_{ij} f_j = g_i.$$

Here $V = \mathbf{C}^n$. Now suppose V is the space of all square integrable functions on the interval $[a, b]$, $L^2([a, b])$, and $k \in L^1([a, b]^2)$. Let $g \in V$, Then we might want to solve the equation

$$\int_a^b k(x, y) f(y) dx = g(x),$$

i.e., if M is ther operator on V,

$$f \mapsto \int_a^b k(x, y) f(y) dx,$$

you might want to solve $Mf = g$.

Examples. 1. Fourier Inversion

Suppose $V = L^2(\mathbf{R})$. Find f such that

$$\int_{-\infty}^{\infty} e^{ixy} f(x) dy = g(x)$$

Solution:

$$f(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ixy} g(x) dx.$$

2. Volterra (ca. 1900)

Suppose $V = C([0, 1])$ and k is the characteristic function of $\{y < x\}$ on $[0, 1] \times [0, 1]$. Let $\lambda \in \mathbf{C}$. Solve the equation:

$$\int_0^1 k(x, y) f(y) - \lambda f(x) = g(x).$$

When $\lambda \neq 0$, $\exists!$ a solution. What about $\lambda = 0$?

One application of the theory is:

Theorem. *(Weiner) Suppose $F(\theta)$ is a function on the unit circle with an absolutely convergent Fourier series. I.e.,*

$$F(\theta) =: \sum_{-\infty}^{\infty} a_n e^{in\theta}$$

and $\sum_{-\infty}^{\infty} |a_n| < \infty$. Then if F has no zeroes, its inverse also has an absolutely convergent Fourier series.

One has an analogue of the determinant when the operator M is "compact;" there is an entire series whose zeroes are the inverses of the non-zero eigenvalues of M (see Grothendieck, La Theorie de Fredholm Bull. Soc. Math. de France, 1956.)

Lecture 2

1. Review of operators on a Banach Space (Chapt III)

Let X be a Banach space over the complex numbers, i.e. a vector space over the complex numbers complete with respect to an absolute value.

By an **operator** on X, I mean a continuous linear transformation $T: X \to X$ and $\mathcal{B}(X)$ will denote the set of all such.

Note $\mathcal{B}(X)$ is a ring. Moreover a linear map $L: X \to X$ is in $\mathcal{B}(X)$ if and only if

$$\mathrm{Sup}_{\substack{x \in X \\ |x| \leq 1}} |L(x)| < \infty.$$

We let $\mathcal{B}^*(X)$ denote the invertible elements of $\mathcal{B}(X)$.

Suppose L is an operator on X.

Open Mapping Theorem. *If L is surjective, then the image of every open set is open.*

The Inverse Mapping Theorem. *If L is bijective then $L \in \mathcal{B}^*(X)$.*

The Closed Graph Theorem. *If $A: X \to X$ is a linear transformation such that the following set (the graph of A) is closed if and only if $A \in \mathcal{B}(X)$:*

$$\{(x, A(x)) \in X \oplus X : x \in X\}.$$

2. The spectrum of an operator.

Let $I \in \mathcal{B}(X)$ be the identity.

Definition. *Let $A \in \mathcal{B}(X)$. Then $\sigma(A)$ is the set of $\lambda \in \mathbf{C}$ such that*

$$A - \lambda I$$

is not invertable. $\sigma(A)$ is called the spectrum of A.

What we said about Volterra's equation can be restated: Let $V = C([0,1])$ and k be the characteristic function of $\{y < x\}$ on $[0,1] \times [0,1]$. Let A be the operator on V

$$A(f) = \int_0^1 k(x,y) f(y).$$

Then our claims imply $\sigma(A) = \{0\}$.

3. Spectra in finite dimensions (review)

Suppose X is finite dimensional. Then $\mathcal{B}(X) =$? If $A \in \mathcal{B}(X)$, what is $\sigma(A)$? If $X \neq 0$, $\sigma(A) \neq \emptyset$. If $\lambda \in \sigma(A)$ there exists a non-zero $u \in X$ such that $Au = \lambda u$. That is u is an eigenvector.

If $\dim X = \infty$ then A need not have any eigenvectors. Eg. Suppose $X = C([0,1])$ and

$$(Af)(x) = x f(x),$$

then A has no eigevectors (why?), but

Theorem (Gelfand)**.** *If $X \neq \emptyset$ and $A \in \mathcal{B}(X)$, $\sigma(A) \neq \emptyset$.*

4. Banach Algebras

Definition. *A Banach algebra (over* **C***) is a* **C***-algebra A (not necessarily with identity) complete with respect to a norm $|| \, ||: A \to [0, \infty)$.*

Examples. 1. The Disk Algebra. 2. $l^1(\mathbf{Z})$). 3. $L^1(\mathbf{R})$. (exercise (for Friday): show associativity, does $L^1(\mathbf{R})$ have an identity?) 4. Any Banach space X. 5. ?

Homework problems due Friday Septemer 6

A. Let X be the Banach space of complex values continuous functions on ther unit interval, $C[0,1]$ with the sup-norm. Set

$$Vf(x) = \int_0^x f(t)dt.$$

(This is the Volterra operator we talked about in class.) (i) Show $V \in \mathcal{B}(X)$ and $||V|| \leq 1$. (ii) Show V is injective and its range is all differentiable functions on $[0,1]$ ($C^1[0,1]$) which vanish at the origin. (iii) Show $\{0\}$ is the spectrum of V.

B. Suppose Y and Z are Banach spaces. Let Y^* denote the set of continuous homomorphisms of Y into **C**. For $h \in Y^*$ set $||h|| = sup_{\substack{y \in Y \\ |y| \leq 1}} |h(y)|$. (i) Show that Y^* is a Banach Space. For $a \in Y \otimes Z$, set

$$||a|| =: inf\left(\sum_{y_1 \otimes z_1 + \cdots + y_n \otimes z_n = a} |y_i| \cdot |z_i| \right),$$

where the infimum is taken over all representations of a as a sum of tensors. (ii) Show $|| \, ||$ is a norm. Let $Y \hat{\otimes} Z$ be the completion $Y \otimes Z$ with respect to this norm. (iii) Show $Y \hat{\otimes} Z$ is a Banach space.

C. An operator is said to have finite rank if its image is a subspace of finite dimension. (i) Show finite rank operators are compact. (ii) Show there is a natural map from $Y^* \hat{\otimes} Y$ into $\mathcal{B}(Y)$ whose image is contained in $\mathcal{B}_0(Y)$.

1. Some more examples of Banach algebras.

Example. 6. Compact operators. Let X be a Banach space. Then a linear map L from X to X on is said to be **compact** if the set $\{L(x) : |x| < 1\}$ has compact closure in X. Let $\mathcal{B}_0(X)$ be the set of compact operators.

Lemma. *$\mathcal{B}_0(X)$ is a closed (two sided) ideal in $\mathcal{B}(X)$.*

Proof. Let $B(a,r) = \{x \in X\colon |x-a| < r\}$. Suppose $T_n \to T$. Suppose $\epsilon > 0$ and $||T - T_n||, \epsilon$ there exist $h_1, \ldots, h_m \in B$ such that

$$T_n(B) \subseteq B(T_n h_i, \epsilon/3).$$

Claim: For any $h \in B(0,1)$ there exists a j such that

$$||Th - Th_j|| < \epsilon. \quad \blacksquare$$

Lemma. *Suppose A is a Banach algebra with an identity 1 ($\neq 0$). Then $||1|| \geq 1$.*

Example. 7. $A = M_n(\mathbf{C})$ and

$$||(a_{i\,j})|| = \sum_{i\,j} |a_{i\,j}|.$$

Example. 8. Suppose G is locally compact group. Eg. $\mathbf{R}$ or $\mathbf{Z}$ under addition or $SL_n(\mathbf{C})$ or the group of as transformations $x \mapsto ax + b$.

Then there is a Haar measure μ on G and we let $L^1(G)$ be the set of all measurable functions on G such that

$$||f|| =: \int_G |f| d\mu < \infty.$$

We define a convolution product on $L^1(G)$,

$$f * g(y) = \int_G f(x) g(yx^{-1}) d\mu(x),$$

2. Quick review of Haar measures

Suppose X is a locally compact Hausdorff space.

Definition. *Suppose $\mathcal{B}$ is the σ-algebra determined by the topology on X. A* **Radon measure** *on X is a countably additive measure $\mu: \mathcal{B} \to [0, \infty]$ such that*
(i) $\mu(E) = sup_{\substack{K \subseteq E \\ K \text{ cpt}}} \mu(K)$
(ii) $\mu(K) < \infty$ for all compact subsets of X.

Theorem (Loomis)**.** *If G is any locally compact group then there exists a Radon measure $\mu \neq 0$ on G unique up to a scalar multiple such that*

$$\mu(xE) = \mu(E) \quad \text{if } x \in G \text{ and } E \in \mathcal{B}.$$

3. The regular representation

Suppose A is a Banach space and an algebra (but not nec. a Banach algebra). For $a \in A$ let L_a be the linear map $A \to A$, $x \mapsto ax$ and let R_a denote the linear map $x \mapsto xa$. We say multiplication on A is **separately continuous** if R_a and L_a are in $\mathcal{B}(A)$ for all $a \in A$

Theorem. *If A has an identity 1_A, and multiplication is separately continuous, then the map $a \mapsto L_a$ is a continuous isomorphism from A onto a Banach subalgebra of $\mathcal{B}(A)$ such that $L_{1_A} = 1$ and $||1_A||^{-1}||x|| \leq ||L_x|| \leq C||x||$.*

Lemma. *If multiplication on A is separately continuous then there exists a constant $C > 0$ such that*

$$||xy|| \leq C||x|| \cdot ||y||.$$

Proof. We have to show $Sup_{|x| \leq 1}||L_x|| < \infty$.
Now use,

PUB (Chapt. III §14)**.** *If $\mathcal{A}$ is a subset of $\mathcal{B}(X)$ such that for all $x \in X$, $Sup\{|Ax|: A \in \mathcal{A}\} < \infty$ then $Sup\{||A||: A \in \mathcal{A}\} < \infty$.*

Homework problems due Friday Septemer 6

A. Let X be the Banach space of complex values continuous functions on ther unit interval, $C[0,1]$ with the sup-norm. Set

$$Vf(x) = \int_0^x f(t)dt.$$

(This is the Volterra operator we talked about in class.) (i) Show $V \in \mathcal{B}(X)$ and $||V|| \leq 1$. (ii) Show V is injective and its range is all differentiable functions on $[0,1]$ ($C^1[0,1]$) which vanish at the origin. (iii) Show $\{0\}$ is the spectrum of V.

B. Suppose Y and Z are Banach spaces. Let Y^* denote the set of continuous homomorphisms of Y into $\mathbf{C}$. For $h \in Y^*$ set $||h|| = sup_{\substack{y \in Y \\ |y| \leq 1}} |h(y)|$. (i) Show that Y^* is a Banach Space. For $a \in Y \otimes Z$, set

$$||a|| =: inf\left(\sum_{y_1 \otimes z_1 + \cdots + y_n \otimes z_n = a} |y_i| \cdot |z_i| \right),$$

where the infimum is taken over all representations of a as a sum of tensors. (ii) Show $||\ ||$ is a norm. Let $Y \hat{\otimes} Z$ be the completion $Y \otimes Z$ with respect to this norm. (iii) Show $Y \hat{\otimes} Z$ is a Banach space.

C. An operator is said to have finite rank if its image is a subspace of finite dimension. (i) Show finite rank operators are compact. (ii) Show there is a natural map from $Y^* \hat{\otimes} Y$ into $\mathcal{B}(Y)$ whose image is contained in $\mathcal{B}_0(Y)$.

Lecture 4

1. Review of the regular representation.

We will call an algebra unital if it has a multiplictive identity. last time we proved,

Theorem. *If A is a unital algebra and a Banach space whose multiplication is seperately continuous, then the map $a \mapsto L_a$ is a homeomorphic isomorphism from A onto a Banach subalgebra of $\mathcal{B}(A)$ such that $L_{1_A} = 1$ and $||1_A||^{-1}||x|| \leq ||L_x|| \leq C||x||$.*

Corollary. *Any Banach space which is also a unital algebra with a seperately continuous multiplication can be renormed so that becomes a Banach algebra whose multiplicative identity has norm 1.*

The map $a \mapsto L_a$ is called the regular representation.

Warning. *In general, if A is not unital, the map $a \mapsto L_a$ is not an isomorphism.*

2. Quick review of Haar measures

Suppose X is a locally compact Hausdorff space.

Definition. *Suppose $\mathcal{B}$ is the Borel σ-algebra of X. A* **Radon measure** *on X is a countably additive measure $\mu: \mathcal{B} \to [0, \infty]$ such that*
(i) $\mu(E) = sup_{\substack{K \subseteq E \\ K \text{ cpt}}} \mu(K)$
(ii) $\mu(K) < \infty$ for all compact subsets of X.

Theorem (Loomis)**.** *If G is any locally compact group then there exists a Radon measure $\mu \neq 0$ on G unique up to a scalar multiple such that*

$$\mu(xE) = \mu(E) \quad \text{if } x \in G \text{ and } E \in \mathcal{B}.$$

(See Loomis, Abstract Harmonic Analysis §29.)
This is called the Haar measure on G.

Banach Algebra Example 8. Suppose G is locally compact group. Eg. $\mathbf{R}$ or $\mathbf{Z}$ under addition or $SL_n(\mathbf{C})$ or the group of as transformations $x \mapsto ax + b$.
Then there is a Haar measure μ on G and we let $L^1(G)$ be the set of all measurable functions on G modulo functions zero almost everywhere such that

$$||f|| =: \int_G |f| d\mu < \infty.$$

We define a convolution product on $L^1(G)$,

$$f * g(y) = \int_G f(x) g(yx^{-1}) d\mu(x),$$

3. The group A^*.

Suppose A is a unital Banach algebra such that $||1_A|| = 1$. We will write 1 for 1_A. (We will suppose the norm of the identity is 1 whenever we talk about unital Banach algebras.)

Theorem. *If $x \in A$ and $||x|| < 1$, then*
(i) $1 - x \in A^*$.
(ii)

$$||(1-x)^{-1}|| \leq \frac{1}{1 - ||x||} \quad \textit{and} \quad ||1 - (1-x)^{-1}|| \leq \frac{||x||}{1 - ||x||}.$$

Corollary 1. *A^* is open in A and the map $x \in A^* \mapsto x^{-1}$ is continuous.*

Corollary 2. *A^* is a topological group with respect to the norm topology.*

Lecture 5

1. A^*

Corollary 2. *A^* is a topological group with respect to the norm topology.*

Definition. *An element $x \in A$ is* **quasi-nilpotent** *if and only if*

$$\lim_{n\to\infty} ||x^n||^{1/n} = 0.$$

Lemma. *If x is quasi-nilpotent, $\sigma(x) \subseteq \{0\}$.*

The Volterra operator is quasi-nilpotent (exercise).

2. Spectrum of an element of a Banach algebra

Suppose A is a unital Banach algebra.

Proposition. *For all $x \in A$, $\sigma(x)$ is a closed subset of $\{\lambda \in \mathbf{C}: |\lambda| \leq ||x||\}$.*

Theorem (Gelfand)**.** *For all $x \in A$, $\sigma(x) \neq \emptyset$.*

Lemma. *If $\lambda_0 \notin \sigma(x)$, then*

$$\lim_{\lambda\to\lambda_0} \frac{1}{\lambda - \lambda_0}\left((x-\lambda)^{-1} - (x-\lambda_0)^{-1}\right) = (x-\lambda)^{-2}.$$

Proof of Theorem.

Suppose $\sigma(x) = \emptyset$. Let $\rho \in A'$ (the set of continuous linear functions from A into $\mathbf{C}$) and set $f(\lambda) = \rho((x-\lambda)^{-1})$. We will show $f = 0$. Then we can appeal to the following Corollary of the Hahn-Banach Theorem,

Theorem. *If V is a subspace of a Banach space X and $h \in V'$, then there exists a $g \in X'$ such that $g|_V = h$ and $|g| = |h|$.* (See Conway, Chapt. II Cor. 6.5)

Now we claim f is entire and $\lim_{\lambda\to\infty} f(\lambda) = 0$.

Corollary 1. *If x is quasi-nilpotent $\sigma(x) = 0$.*

Corollary 2. *The only Banach division algebra over* $\mathbf{C}$ *is* $\mathbf{C}$.

(The only finite dimensional division algebra over $\mathbf{C}$ is $\mathbf{C}$ (exercise).)

2. The Spectral Radius

Suppose A is a unital Banach algebra.

Definition. *Suppose $x \in A$. Then set $r(x) = Sup\{|\lambda|: \lambda \in \sigma(x)\}$, This is the* **spectral radius** *of x.*

Remarks. *1. $r(x) \leq ||x||$.*
2. $r(\lambda x) = |\lambda| r(x)$.
3. If $P(x)$ is a polynomial and $\lambda \in \sigma(x)$ then $P(\lambda) \in \sigma(P(x))$.
4. $r(x) \leq \liminf_{n \geq 1} ||x^n||^{1/n}$.

Theorem (Gelfand-Mazur)**.** *For all $x \in A$,*

$$r(x) = \lim_{n \to \infty} ||x^n||^{1/n}.$$

Claim. *Fix $x \in A$, $|\lambda| < 1/r(x)$. Then $\{(\lambda x)^n : n \in \mathbf{Z}\}$ is bounded.*

Suppose we knew this, then

$$|\lambda|^n ||x^n|| \leq M < \infty \quad \text{so} \quad ||x^n||^{1/n} \leq M^{1/n} \frac{1}{|\lambda|} \quad \text{if } \lambda \neq 0.$$

Homework problems due Friday, September 13

1. Let V be the Volterra operator on $C[01]$. (a) Show V is quasinilpotent. (Hint: Give an explicit formula for V^n.) (b) Deduce that $\sigma(V) = 0$.
2. Show the only (non-zero) finite deimensional division algebra over $\mathbf{C}$ is $\mathbf{C}$
3. Let X be a compact Hausdorff Space and $A = C(X)$ (the Banach algebra of continuous complex valued functions). (a) Show that $\sigma(f) = f(X)$. (b) Deduce that $r(x) = ||x||$ in this Banach algebra.
4. Let H be a Hilbert space and suppose $\{e_1, e_2, \ldots\}$ is an orthonormal basis for H. Suppose $a_1, a_2, \ldots$ is a sequence of complex number. (a) Show there is an $A \in \mathcal{B}(H)$ such that $A\, e_n = a_n e_n$, $\forall n \geq 1$. (b) Show that A is quasi-nilpotent.

Lecture 6

1. Homework problems

Correction to homework problem for this week. Problem 4. should be:

Let H be a Hilbert space and suppose $\{e_1, e_2, \ldots\}$ is an orthonormal basis for H. Suppose $a_1, a_2, \ldots$ is a sequence of complex numbers such that $a_n \to 0$. (a) Show there is an $A \in \mathcal{B}(H)$ such that $A\,e_n = a_n e_{n+1}$, $\forall n \geq 1$. (b) Show that A is quasi-nilpotent. To solve B (i): Suppose Y and Z are Banach spaces. For $a \in Y \otimes Z$, set

$$||a|| =: inf\Bigg(\sum_{y_1\otimes z_1+\cdots+y_n\otimes z_n=a} |y_i|\cdot|z_i|\Bigg),$$

where the infimum is taken over all representations of a as a sum of tensors. Show $||\ ||$ is a norm. Suppose $||a|| = 0$. We want to show $a = 0$. We will use the Hahn-Banach Theorem.

Suppose $A\colon Y \otimes Z \to \mathbf{C}$ is a continuous linear map w.r.t. $||\ ||$, then, claim,

$$||A|| = \max_{||y||\leq 1, ||z||\leq 1} ||A(y\otimes z)|| := M.$$

It is clear that $M \leq ||A||$. Also, $||A(y \otimes z)|| \leq ||y|| \cdot ||z|| M$.

2. The Gelfand-Mazur Theorem

Suppose A is a unital Banach algebra.

Remarks. *1. If $P(x)$ is a polynomial and $\lambda \in \sigma(x)$ then $P(\lambda) \in \sigma(P(x))$.*
2. $r(x) \leq \liminf_{n\geq 1} ||x^n||^{1/n}$.

Theorem (Gelfand-Mazur)**.** *For all $x \in A$,*

$$r(x) = \lim_{n\to\infty} ||x^n||^{1/n}.$$

Claim. *Fix $x \in A$, $|\lambda| < 1/r(x)$. Then $\{(\lambda x)^n : n \in \mathbf{Z}\}$ is bounded.*

Suppose we knew this, then

$$|\lambda|^n ||x^n|| \leq M < \infty \quad \text{so} \quad ||x^n||^{1/n} \leq M^{1/n}\frac{1}{|\lambda|} \quad \text{if } \lambda \neq 0.$$

Proof of Claim.

By the PUB theorem (see Chapt. III Cor. 14.3), it suffices to show for every $\rho \in A'$

$$\sup_{n\geq 1} |\rho(\lambda^n x^n)| < \infty.$$

Consider the function on the disk $B(0, 1/r(x))$ around the origin of radius $1/r(x)$ in $\mathbf{C}$,

$$f(\lambda) = \rho((1-\lambda x)^{-1}).$$

But (i) f is analytic and (ii) on disk $B(0, 1/||x||)$,

$$f(\lambda) = \sum_{n=0}^{\infty} \lambda^n \rho(x^n).$$

Lecture 7

1. Examples of operators

1. *Multiplication operators*

Let (S, μ) be a σ-finite measure space. Let $X = L^p(S, \mu)$.

Definition. *Suppose $f \in L^\infty(S, \mu)$. Let $M_f\colon X \to X$ be $x \mapsto fx$.*

Proposition. *(i) M_f is continuous,*
(ii) $||M_f|| = ||f||_\infty$
(iii) $r(M_f) = ||M_f||$.
Moreover the map $f \mapsto M_f$ is an isometric isomorphism of L^∞ into $\mathcal{B}(X)$.

Proof. "Clearly," $||M_f|| \leq ||f_\infty||$. Now $(L^1)' = L^\infty$ because S is σ-finite (see Dunford and Schwartz, IV.8.5). This implies the the image of the unit ball in L^1 is weak* dense in the unit ball in $(L^\infty)'$ (Conway V Prop. 41) and so there exists a sequence $k_n \in L^1$ such that $||k_n||_1 \leq 1$ and

$$|\langle k_n, f\rangle| \to ||f||_\infty.$$

Suppose $\frac{1}{p} + \frac{1}{q} = 1$ and

$$h_n = \frac{k_n}{|k_n|}|k_n|^{1/p} \quad \text{and} \quad g_n = |k_n|^{1/q}$$

Then $||g_n||_q \leq 1$, $||h_n||_p \leq 1$ and

$$|\langle M_f h_n, g_n\rangle| \leq ||M_f||;$$

while $|\langle M_f h_n, g_n\rangle| \to ||f||_\infty$. Thus, $||M_f||^n = ||M_f^n||$.

Suppose $S = [0, 1]$, $\mu = dt$ and $f = e^{2\pi i x}$. Then $\sigma(M_f) = S^1$ the unit circle.

2. *Weighted shifts on l^2.*

Suppose H is a Hilbert space with orthonormal basis $\{e_1, e_2 \ldots\}$. Suppose $\{\alpha_i\}$ is a bounded sequence in $\mathbf{C}$. Claim: $\exists!\, A \in \mathcal{B}(H)$ such that

$$Ae_i = a_a e_{i+1}.$$

The spectrum of A is a union of circles around the origin.

Suppose $a_i = 1$ for all i. Then $\sigma(A) = B[0, 1]$. Let A^* be the adjoint of A.

1. The operator $M_{e^{2\pi it}}$

What we need is:

Spectral Permanence Theorem. *Suppose B is a complete Banach subalgebra of a Banach algebra A and $x \in B$. Then,*

$$\partial\sigma_B(x) \subseteq \sigma_A(x).$$

2. Nuclear operators

Definition. *Let X be a Banach space. By a* **nuclear** *operator, we mean an operator in the image of $X'\hat{\otimes}X$ in $\mathcal{B}(X)$.*

Suppose (M, μ) is a locally compact space with a measure.

Proposition (Grothendieck)**.** *Suppose M is compact and $K(s,t)$ a continuous function on $M \times M$. Then the operator*

$$I_K\colon f \mapsto \int K(s,t)f(T)d\mu(t).$$

is nuclear with norm equal to

$$\int \sup_s |K(s,t)|d|\mu|(t).$$

Definition. *Suppose E is a Banach space. Then $L^1_E(M,\mu)$ is the Banach space of $L^1(\mu)$ functions (or rather classes of functions) on M with values in E with norm, $|| \, ||'$,*

$$||f||' = \int |||f(s)||d|\mu|(s)$$

Theorem (Grothendieck)**.** *The natural map from $L^1(M,\mu)\hat{\otimes}E$ into $L^1_E(M,\mu)$ is an isometry.*

Proof. (Exercise.)

The proposition I stated in the notes last time is false. Please delete it from your files. The correct statement is:

Corollary. *Suppose E is a Banach space. There is a natural injection T_μ from $L^1_E(M,\mu)$ into $C_0(M)'\hat{\otimes}E$ such that if $\varphi \in C_0(M)$, $f \in L^1_E(M,\mu)$,*

$$\int \varphi(s)f(s)d\mu(s) = T_\mu(f)(\varphi).$$

Moreover, the norm of the linear map $\varphi \mapsto T_\mu(f)(\varphi)$ from $C_0(M)$ into E is $\int ||f(s)||d|\mu|(s)$. Moreover, $C_0(M)'\hat{\otimes}E = \bigcup_\nu Im(T_\nu)$ as ν ranges over all measures on M.

Reason. Given $f \in L^1(N,\mu)$ we get by the theorem an element of $L^1(M,\mu)\hat{\otimes}E$ and there is a natural map of $L^1(M,\mu)$ into $C_0(M)'$. Expliciting this yields the first part of the corollary (exercise).

Now suppose the hypothees of the first proposition. Apply this corollary to $E = C(M)$ and $f(s) = (t \mapsto K(s,t))$.

3. Volterra on $L^p[0,1]$

For $1 \le p < \infty$, $f \in L^p[0,1]$ let $Vf(x) = \int_0^x f(t)dt$.

1. $||V|| \le 1$ (when $p = 2$, $||V|| = 2/\pi$ (exercise)).
2. $V^n f(x) = \frac{1}{(n-1)!}\int_0^x (x-t)^{-1} f(t)dt$.
3. Claim: If $\varphi \in L^1[0,1]$ and

$$C_\varphi f(x) = \int_0^x \varphi(x-t)f(t)dt,$$

then $||C_\varphi f||_p \le ||\varphi||_1 ||f||_p$.

Proof. Choose $g \in L^q$, $g \ge 0$ and $||g||_q \le 1$.

Hence, $||V^n|| \le \frac{1}{n!}$ so is quasi-unipotent.

Homework problems due Friday Sept. 20

1. Let $\{e_1, e_2, \ldots\}$ be an orthonormal basis for a Hilbert space H. Suppose $\alpha_1, \alpha_2, \ldots$ is as bonded sequence of complex numbers. Let $A \in \mathcal{B}(H)$ be the weighted shift defined by

$$Ae_n = \alpha_n e_{n+1}, \quad n = 1, 2, \ldots$$

a) Show that for every $\lambda \in \mathbf{C}$, $|\lambda| = 1$, there is a unitary operator U_λ such that $U_\lambda A U_\lambda^{-1} = \lambda A$.

b) Deduce that the Spectrum of A is a union of circles centered at ther origin.

2. Suppose x is an element of a unital Banach algebra A whose spectrum is the unit circle S^1. Show that for any circle T in the complex plane, there is an element of A whose spectrum is T.

3. Let A be a unital Banach algebra and let M_n be the set of $n \times n$ matrices over $\mathbf{C}$. Let ω be a surjective homomorphism from A onto M_n.

a) Show that the kernel of ω is maximal.

b) Show that all maximal ideals are closed.

b) Deduce that ω is continuous if the topology on M_n is determined by some norm.

4. Prove Grothendieck's theorem about $L^1_E(M, \mu)$.

1. Volterra on $L^p[0,1]$

For $1 \le p < \infty$, $f \in L^p[0,1]$ let $Vf(x) = \int_0^x f(t)dt$.

1. $||V|| \le 1$ (when $p = 2$, $||V|| = 2/\pi$ (exercise)).

2. $V^n f(x) = \frac{1}{(n-1)!}\int_0^x (x-t)^{n-1} f(t)dt$.

Claim: If $\varphi \in L^1[0,1]$ and

$$C_\varphi f(x) = \int_0^x \varphi(x-t)f(t)dt,$$

then $||C_\varphi f||_p \le ||\varphi||_1 ||f||_p$.

Proof. Choose $g \in L^q$, $g \ge 0$ and $||g||_q \le 1$.

Hence, $||V^n|| \le \frac{1}{n!}$ so is quasi-unipotent.

2. Ideals in a Banach Algebra

Suppose A is a Banach algebra (not necessarily) with a unit.

For us by an ideal I in A will be a two sided ideal, i.e. $AI \subseteq I \supseteq IA$.

Example. *Suppose X is a Banach space, then the sets of nuclear or compact operators are ideals in $\mathcal{B}(X)$ (exercise).*

If I is an ideal of A, then A/I is a quotient ring and the sequence

$$0 \to I \to A \to A/I \to 0$$

is exact. Moreover, A/I is naturally a Banach algebra such that $A \to A/I$ is continuous if I is closed.

Remark. *The closure of an ideal is an ideal.*

Lemma. *Let A be a unital Banach algebra and I an ideal in A. Then the following are equivalent:*

(i) $I \neq A$

(ii) $1 \notin A$,

(iii) $1 \notin \bar{I}$

(iv) $||1 - z|| \geq 1$ *for all* $z \in I$.

Corollary. *If A is unital, maximal ideals are closed.*

3. Homomorphisms

Suppose $\omega: A \to B$ is a continuous homomorphism of Banach algebras. Then we get an injection

$$A/Ker(\omega) \xrightarrow{\tilde{\omega}} B.$$

Proposition. *$\tilde{\omega}$ is continuous, $Ker(\omega)$ is closed and*

$$||\tilde{\omega}|| \leq ||\omega||.$$

Remark. *If I is a maximal ideal, then A/I is simple (i.e. has no two sided ideals besides 0 and A). If A is unital and communative then you know what A/I is.*

3. Commutative Banach Algebras (Gelfand Theory)

Suppose A is a commutative unital Banach algebra

Definition. *The spectrum of A is $\sigma(A) = Hom_{\mathbf{C}}(A, \mathbf{C})$. (For us a homomorphism of unital rings will take 1 to 1.)*

The theorem we are going after is:

Theorem (Gelfand)**.** *If $x \in A$,*

$$\sigma(x) = \{\omega(x) : \omega \in \sigma(A)\}.$$

Proposition. *If $\omega \in \sigma(A)$, ω is continuous. (ii) $\sigma(A) \Leftrightarrow$ maximal ideals of A. (iii) If $\omega \in \sigma(A)$, and $x \in A$,*

$$|\omega(x)| \leq r(x).$$

Corollary. *If $\omega \in \sigma(A)$, $|\omega| = 1$.*

Note: $\sigma(A) \neq \emptyset$.

Lecture 10

0. Some measure theory.

To prove Grothendieck's Theorem on $L^1_E(X,\mu)$ all one needs is:

Corollary III.3.8 (Dunford and Schwartz)**.** *Suppose (X,μ) is a measure space, F is a Banach space and $1 \le p < \infty$. Then the set of μ-simple, μ integrable functions with values in F is dense in $L^p_F(XC,\mu)$.*

1. Homomorphisms revisited

Suppose $\omega\colon A \to B$ is a continuous homomorphism of Banach algebras. Then we get an injection

$$A/Ker(\omega) \xrightarrow{\tilde{\omega}} B.$$

Proposition. *$Ker(\omega)$ is closed, $\tilde{\omega}$ is continuous, and*

$$||\tilde{\omega}|| \le ||\omega||.$$

2. Commutative Banach Algebras (Gelfand Theory)

Suppose A is a commutative unital Banach algebra

Definition. *The spectrum of A is $\sigma(A) = Hom_{\mathbf{C}}(A,\mathbf{C})$. (For us a homomorphism of unital rings will take 1 to 1.)*

Proposition. *If $\omega \in \sigma(A)$, (i) ω is continuous. (ii) $\sigma(A) \Leftrightarrow$ maximal ideals of A. (iii) If $\omega \in \sigma(A)$, and $x \in A$,*

$$|\omega(x)| \le r(x).$$

Corollary. *If $\omega \in \sigma(A)$, $|\omega| = 1$.*

Note: $\sigma(A) \neq \emptyset$.

3. The topology on the Spectrum of A

Some recollections on weak topologies*

If X is a Banach space, a basis of open neighborhoods of the origin in X' for the weak* topology is:

$$\{B(S,\epsilon) =: \{h \in X' \colon |h(x)| < \epsilon\}\}$$

where S is a finite subset of X. It follows that if $\{h_n\}$ is a sequence of elements in X',

$$h_n \to 0 \text{ (weakly)} \quad \text{if and only if} \quad h_n(x) \to 0$$

for all $x \in X$.

Remarks. *1.* $\sigma(A) \subseteq ball(A')$.

2. $\sigma(A)$ *is weak* closed.*

Proof. Suppose $f \in A'$. Then $f \in \sigma(A)$ if and only if (i) $f(xy) - f(x)f(y) = 0$, $\forall x, y \in A$. ∎
and (ii) $f(1) = 1$.

As a consequence of this and Alaoglu's Thm. that $ball(A')$ is compact in the weak* topology (Conway V.3.1), with the weak* topology $\sigma(A)$ is a compact Hausdorff space.

4. The Gelfand Transform

For $x \in A$ and $\omega \in \sigma(A)$, set $\hat{x}(\omega) = \omega(x)$.

Proposition. *The Gelfand Transform is a homomorphism of* A *into* $C(\sigma(A))$ *such that (i)* $1 \in \hat{A}$, *(ii)* $\hat{A}$ *separates points, (iii)* $||\hat{x}||_\infty \leq r(x) \leq ||x||$ *for all* $x \in A$.

Lecture 11

1. The Gelfand Transform

Suppose A is a unital commutative Banach algebra. For $x \in A$ and $\omega \in \sigma(A)$, set $\hat{x}(\omega) = \omega(x)$.

Proposition. *The Gelfand Transform is a homomorphism of A into $C(\sigma(A))$ such that (i) $1 \in \hat{A}$, (ii) $\hat{A}$ separates points, (iii) $||\hat{x}||_\infty \leq r(x) \leq ||x||$ for all $x \in A$ and*

Theorem (Gelfand)**.** *If A is a unital commutative Banach algebra. Then for all $x \in A$,*

$$\sigma(x) = \{\hat{x}(p) \colon p \in \sigma(A)\}.$$

Corollary. *Suppose $x \in A$, then $x \in A^{-1}$ if and only if $\hat{x} \in C(\sigma(A))^{-1}$.*

Corollary. *The kernel of the Gelfand transform is $\{x \in A \colon \sigma(x) = \{0\}\}$.*

This theorem also yields another proof, in the commutative case, that the spectrum of an element of a Banach algebra is not empty.

2. $l^1(\mathbf{Z})$

$A =: l^1(\mathbf{Z})$ under convolution.

For $\lambda \in S^1$, define $\omega_\lambda \in l^1(\mathbf{Z})'$ by

$$\omega_\lambda(a) = \sum_n a_n \lambda^n,$$

for $a = (a_n) \in l^1(Z)$. Set $\zeta = \chi_{\{1\}} \in l^1(\ z)$. Remarks: (i) $\zeta \in A^{-1}$

(ii) $||\zeta^n|| = 1$

(iii) $a = \sum_{n=-\infty}^{\infty} a_n \zeta^n$, $\forall a = (a_n) \in l^1(\mathbf{Z})$.

(iv) $\omega_\lambda \in \sigma(A)$.

Proposition. *The map $\lambda \mapsto \omega_\lambda$ from S^1 to $\sigma(A)$ is a homeomorphism.*

Conclusion. *If $A = l^1(\mathbf{Z})$ then $\hat{A}$ is the algebra of all functions on S^1 which have absolutely convergent Fourier series.*

Is the map $A \to \hat{A}$ is an isomorphism in this case?

3. Wiener's Tauberian Theorem

Let AC be the ring of absolutely convergent Fourier series.

Theorem (Wiener)**.** *Let $f \in AC \subseteq C(S^1)$ and suppose that f never vanishes. Then $f^{-1} \in AC$.*

Gelfand's proof: $AC = \{\hat{x} : x \in l^1(\mathbf{Z})\}$.

4. $A =: C(X)$

Suppose X is a compact Hausdorf space. For $p \in X$ and $f \in A$, set $\omega_p(f) = f(p)$.

Theorem. *The map $p \mapsto \omega_p$ is an isomorphism of X onto $\sigma(C(X))$ and*

$$\hat{f}(\omega_p) = f(p).$$

Definition. *If Y is a bounded subset of $\mathbf{C}$, let*

$$c(Y) = \bigcap \{D : D \subseteq Y,\ D \text{ is a closed disk}\}.$$

It will be enough to prove:

Claim. *If $\omega \in \sigma(C(X))$ and $f_1, \ldots, f_k$ are real valued functions in $C(X)$, the there exists a $p \in X$ such that*

$$\omega(f_i) = f_i(p).$$

To prove this we will use Riesz-Marcov and

Lemma. *Let $\rho \in A'$. Then the following are equivalent:*
(i) $\rho(f) \in c(f(X)), \forall f \in C(X)$
(ii) $||\rho|| = \rho(1) = 1$.

Corollary. $||\rho|| = \rho(1) = 1$ *implies* $\rho(\bar{f}) = \overline{\rho(f)}$.

Homework problems due Sept. 27

1. A unital Banach Algebra A is said to be **singly-generated** if there is an element $a \in A$ such that $\{1, a\}$ generates A as a Banach algebra (i.e., if the completion of the subalgebra of A generated by 1 and a is A). Show that the spectrum of a singly generated Banach algebra is homeomorphic to a compact subspace of $\mathbf{C}$.

2. Let $A(\Delta)$ be the **disk algebra** where Δ is the closed unit disk in $\mathbf{C}$. Let $f \in A(\Delta)$ and let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad |z| < 1$$

be the power series expansion of f. Suppose

(i) $f(z) \neq 0$, for all $|z| \leq 1$. (ii) $\sum_{n=0}^{\infty} |a_n| < \infty$. Show that if the power series expansion of $1/f(z)$ is $\sum_{n=0}^{\infty} b_n z^n$, then $\sum_{n=0}^{\infty} |b_n| < \infty$.

3. Show that the Gelfand transform for $A = l^1(\mathbf{Z})$ is an injection and its image is not dense in $C(\sigma(A))$.

Lecture 12

1. $A =: C(X)$

Suppose X is a compact Hausdorff space. For $p \in X$ and $f \in A$, set $\omega_p(f) = f(p)$.

Theorem. *The map $p \mapsto \omega_p$ is a homeomorphism of X onto $\sigma(C(X))$.*

It will be enough to prove:

Claim. *If $\omega \in \sigma(C(X))$ and $f_1, \ldots, f_k$ in $C(X)$, the there exists a $p \in X$ such that*

$$\omega(f_i) = f_i(p).$$

To prove this we will use Riesz-Marcov.

2. The Riez-Marcov Theorem

In this section, suppose X is a locally compact Hausdorff space. Let $C_c(X)$ be the linear space of compactly supported continuous functions on X. A linear functional on $C_c(X)$ is said to be positive if for all real valued $f \in C_c(X)$ such that $f(x) \geq 0, \forall x \in X$,

$$\Lambda(f) \in \mathbf{R} \text{ and is } \geq 0.$$

(Note: If X is compact, $C_c(X) = C(X)$ and positive linear functionals are continuous.) Also a **Radon Measure** is a positive inner regular Borel measure finite on compact sets.

Theorem (Riez-Marcov)**.** *Let Λ be a positive linear functional on $C_c(X)$. Then there exists a unique Radon measure μ such that*

$$\Lambda(f) = \int_X f d\mu.$$

This gives a bijection between positive linear functionals for all opens $U \subseteq X$,

$$\mu(U) = \sup\{\Lambda(f) \colon f \in C_c(X), supp(f) \subseteq U, 0 \leq f \leq 1\}.$$

3. End of Proof

Return to the notation and hypotheses of §1.

Definition. *If Y is a bounded subset of $\mathbf{C}$, let*

$$c(Y) = \bigcap \{D \colon D \subseteq Y,\ D \text{ is a closed disk}\}.$$

Lemma. *Let $\rho \in A'$. Then the following are equivalent:*

(i) $\rho(f) \in c(f(X)), \forall f \in C(X)$

(ii) $||\rho|| = \rho(1) = 1$.

Corollary. $||\rho|| = \rho(1) = 1$ *implies* $\rho(\bar{f}) = \overline{\rho(f)}$.

Corollary. *There exists a Radon measure on X such that for all $f \in C(X)$*

$$\omega(f) = \int f d\mu.$$

Now set,

$$g(p) = \sum_{i=1}^{k} |f_i(p) - \omega(f_i)|^2.$$

Lecture 13

1. The radical of a commutative Banach algebra

Suppose A is unital commutative Banach algebra.

We have defined the Gelfand transform $G: A \to \hat{A} \subseteq C(\sigma(A))$. The kernel of G is called the **radical** of A, denoted $rad(A)$. We know

Proposition. *$rad(A)$ is the intersection of the maximal ideals of A.*

Definition. *The ring A is* **semisimple** *if $rad(A) = \{0\}$ and called* **radical** *if $A/rad(A) \cong \mathbf{C}$.*

Examples. (i) If X is compact and $A = C(X)$.

(ii) $A = l^1(\mathbf{Z})$

(iii) V is the Volterra operator on $L^2[0,1]$ and A is the completion of the subalgebra of $\mathcal{B}(L^2[0,1])$ generated by 1 and V.

In general, $A/rad(A)$ is semi-simple.

2. Weighted convolution algebras

Let $(w) = \{w_0, w_1, \ldots\}$ be positive real numbers such that

(i) $w_0 = 1$

(ii) $w_{m+n} \le w_m w_n$.

Examples. $w_n = r^n$ or $w_n = \frac{1}{n!}$.

Definition. *We define $l^1(\mathbf{Z}^+, w)$ to be the Banach algerbra consiting of sequences $a := (a_n)$ of complex numbers such that*

$$||a|| := \sum_{n \ge 0} |a_n| w_n < \infty.$$

Moreover, if $b := (b_n) \in l^1(\mathbf{Z}^+, w)$, we set

$$(a * b)_n = \sum_{i+j=n} a_i b_j.$$

Let $A = l^1(\mathbf{Z}^+, w)$.

Lemma. *$(A, || \, ||)$ is a commutative unital Banach algebra.*

Lemma. *Let $\omega \in \sigma(A)$ and $\lambda = \omega(\chi_{\{1\}}$. Then if $a = (a_n)$,*

$$\omega(a) = \sum_{n \geq 0} a_n \lambda^n$$

Proposition. *$\sigma(A)$ is isomorphic to the closed disk of radius* $\inf w_n^{1/n}$.

Examples. If $w_n = r^n$, then A is semi-simple. Also,

$$\sigma(a_n) = \{\sum_{n \geq 0} a_n \lambda^n \colon |\lambda| \leq r\}.$$

If $w_n = \frac{1}{n!}$, then A is radical.

3. Relative Spectra

Suppose A is a unital (not necessarily commutative) Banach algebra and B is a closed unital subalgebra.

How does the spectrum of an element x of B relative to B compare to the spectrum of x relative to A?

We know $\sigma_B(x) \supseteq \sigma_A(x)$.

Example. $A = C(S^1)$ and B, the closure of the subring P of polynomials. We know $\sigma(A) = S^1$. Claim: $\sigma(B) = B[0, 1]$.

Lecture 14

1. The Spectral Permanence Theorem

Definition. *Suppose X is a compact subset of $\mathbf{C}$. By a* **hole** *of X, we mean a bounded component of $\mathbf{C}\backslash X$.*

Now suppose $B \subseteq A$ are unital Banach algebras. Then,

Proposition. *If $x \in B$,*

$$\sigma_B(x)\backslash\sigma_A(x) = \bigcup_i H_i,$$

where the H_i are some of the holes of $\sigma_A(x)$.

This is based on,

The Spectral Permanence Theorem. *Let $1 \in B \subseteq A$ be as above. Then,*

$$\partial\sigma_B(x) \subseteq \sigma_A(x).$$

Proof. Suppose $0 \in \partial\sigma_B(x)$ and $0 \notin \sigma_A(x)$. Then $\exists\lambda_n \in \mathbf{C}\backslash\sigma(x)$, $\lambda_n \to 0$.

Corollary. *Suppose H is a hole of $\sigma_A(x)$. Then either $H \subseteq \sigma_B(x)$ or $H \cap \sigma_B(x) = \emptyset$.*

Proof. Let $X = \sigma_B(x) \cap H$. Then

$$\partial X = \partial\sigma_B(x) \cap H \subseteq \sigma_A(x) \subseteq \mathbf{C}\backslash H.$$

Corollary. *Let $x \in A$ and H_∞ the unbounded component of $\mathbf{C}\backslash\sigma_A(x)$. Then if $\lambda \in H_\infty$, there exist polynomials $p_1(z), p_2(z), \ldots$ such that in A*

$$p_n(x) \to (x - \lambda)^{-1}.$$

In fact,

Proposition. *Suppose X is a compact subset of $\mathbf{C}$ and $P(X)$ is the completion of the subring of $C(X)$ generated by polynomials. Then $\sigma(P(X))$ equals X union all the holes of X.*

Homework problems due Oct. 4

1. Suppose H is a continuous function on $[0,1] \times [0,1]$. Show that

$$f \mapsto \int_0^1 K(x,y) f(y) dy$$

is a continuous operator on $L^2[0,1]$ and the collection of such operators is a Banach subalgebra of $\mathcal{B}(L^2[0,1])$. Is it unital? Commutative?

2. Let X be a compact subset of $\mathbf{C}$ and let $R(X)$ be the norm closure of the rational functions with poles in $\mathbf{C}\backslash X$. Show that $\sigma(R(X))$ may be naturally identified with X.

Let $(H, \langle \ , \ \rangle)$ be a Hilbert space.

3. Let T be a bounded operator on $\mathcal{B}(H)$. Show that

$$\overline{ran(T)} = (ker T^*)^{\perp}.$$

4. For $T \in \mathcal{B}(H)$, define the **numerical radius** of T, $w(T)$, by

$$w(T) = \sup_{||x||=1} |\langle Tx, x\rangle|.$$

Show

$$\frac{1}{2}||T|| \leq w(T) \leq ||T||.$$

5. For a finite dimensional vector space V over $\mathbf{C}$, show that there is a natural map from $V' \otimes V$ to $\mathbf{C}$ which takes $h \otimes v$ to $h(v)$. (ii) Show that when one identifies $V' \otimes V$ with linear maps from V to itself, this is the trace. (iii) If X is a Banach space, show that this map extends to $X' \hat{\otimes} X$.

Lecture 15

1. End of the Spectral Permanence Theorem

The Spectral Permanence Theorem. *Suppose $B \subseteq A$ are unital Banach algebras. Then,*

$$\partial\sigma_B(x) \subseteq \sigma_A(x).$$

Corollary. *Then, if $x \in B$, $\sigma_B(x)\backslash\sigma_A(x) = \bigcup_i H_i$, where the H_i are some of the holes of $\sigma_A(x)$.*

Corollary. *Let $x \in A$ and H_∞ the unbounded component of $\mathbf{C}\backslash\sigma_A(x)$. Then if $\lambda \in H_\infty$, there exist polynomials $p_1(z), p_2(z), \ldots$ such that, in A, $p_n(x) \to (x-\lambda)^{-1}$.*

Proposition. *Suppose X is a compact subset of $\mathbf{C}$ and $P(X)$ is the completion of the subring of $C(X)$ generated by polynomials. Then $\sigma(P(X))$ equals X union all the holes of X.*

2. Operators on a Hilbert Space

Suppose $(H, \langle\ ,\ \rangle)$ is a Hilbert space. Recall (Conway Thm. 1.3.4),

Riesz Representation Theorem. *If $L \in H'$, $\exists!\ h_0 \in H$ such that*

$$L(h) = \langle h, h_0\rangle.$$

Moreover, $||L|| = ||h_0||$.

Proposition. *For all $L \in \mathcal{B}(H)$, $\exists!\ L^* \in \mathcal{B}(H)$ such that*

$$\langle Lx, y\rangle = \langle x, L^*y\rangle.$$

Definition. *A sesquilinear form $[\ ,\ ]$ is* **bounded** *on H if and only if $\exists C \in \mathbf{R}$ such that*

$$||[x,y]|| \leq C||x|| \cdot ||y||.$$

Lemma (Riesz)**.** *For each bounded sesquilinear forms* $[\ ,\ ]$ *on* H $\exists!\ T \in \mathcal{B}(H)$ *such that*

$$[x, y] = \langle Tx, y\rangle.$$

Proof. Consider $L_x := y \mapsto \overline{[x, y]}$.

Proof of Proposition.

Look at $[x, y] = \overline{\langle Lx, y\rangle}$.

3. Properties of the Adjoint Map

(i) $A^{**} = A$

(ii) $(aA + bB)^* = \bar{a}A^* + \bar{b}B^*$

(iii) $(AB)^* = B^*A^*$

(iv) $||A^*A|| = ||A||^2$.

By an **involutive** Banach algebra we mean a Banach algebra with an involution $a \mapsto a^*$ satisfying (i)-(iii).

Examples. (i) $l^1(\mathbf{Z})$. (ii) Convergent power series on $B[0, 1]$. (iii) $C(X)$.

4. C^*-algebras

Definition. *A* C^**-algebra is an involutive Banach algebra such that* $||a^*a|| = ||a||^2$.

Examples. Let H be a Hilbert space. Then any norm closed subalgebra A of $\mathcal{B}(H)$ such that $A^* = A$.

What about $l^1(\mathbf{Z})$?

Theorem. *If* A *is a commutative unital* C^**-algebra then the Gelfand transform is isometric.*

Lecture 16

Problem: Suppose f is analytic on the open disk $B(0,1)$ and extends to a continuous function on the closed disk $B[0,1]$. Does f extend to an analytic function on a neiborhod of this closed disk?

1. The Adjoint Map

Suppose $(H, \langle\ ,\ \rangle)$ is a Hilbert space.

Recall, we showed, for all $L \in \mathcal{B}(H)$, $\exists!\ L^* \in \mathcal{B}(H)$ such that $\langle Lx, y\rangle = \langle x, L^*y\rangle$.

The following arer true:

(i) $A^{**} = A$

(ii) $(aA + bB)^* = \bar{a}A^* + \bar{b}B^*$

(iii) $(AB)^* = B^*A^*$

(iv) $||A^*A|| = ||A||^2$.

By an **involutive** Banach algebra we mean a Banach algebra with an involution $a \mapsto a^*$ satisfying (i)-(iii).

Examples. (i) $l^1(\mathbf{Z})$. (ii) Convergent power series on $B[0,1]$. (iii) $C(X)$.

2. C^*-algebras

Definition. *A C^*-algebra is an involutive Banach algebra A such that*

$$||a^*a|| = ||a||^2.$$

Lemma. *(i) $||A^*|| = ||A||$ so that $A \mapsto A^*$ is continuous. (ii) If A is unital* $||1|| = 1$.

Examples. (i) Let H be a Hilbert space. Then any norm closed subalgebra A of $\mathcal{B}(H)$ such that $A^* = A$. (ii) Eg. Suppose $T \in \mathcal{B}(H)$. Eg. Let $C^*(T)$ be the smal lest Banach subalgebra generated by $1, T$ and T^*. (iii) $C(X)$ for X compact. (iv) What about $l^1(\mathbf{Z})$?

3. The Exponential Map

Suppose A is a unital Banach algebra. For $a \in A$, set

$$exp(a) = \sum_{n=0}^{\infty} \frac{a^n}{n!}$$

Proposition. *If $x, y \in A$ such that $xy = yx$, then*

$$\exp(x + y) = \exp(x)\exp(y).$$

Lemma. *Suppose $a_n b_n \in A$ such that $\sum_n ||a_n||$ and $\sum_{n=0} ||b_n||$ are finite. Then,*

$$\Big(\sum_{n=0}^{\infty} a_n\Big)\Big(\sum_{n=0}^{\infty} b_n\Big) = \sum_{n=0}^{\infty} c_n,$$

where $c_n = \sum_{i+j=n} a_i b_j$. Moreover, $\sum_{n=0}^{\infty} ||c_n|| < \infty$.

Example. The series $\sum_0^\infty \frac{x^n}{n^2}$ has radius of convergence 1 and converges on the unit circle. Now a find an example which is not absolutely convergent.

1. Commutative C^*-algebras

Theorem. *If A is a commutative unital C^*-algebra, then the Gelfand transform is an $*$-isometry of A onto $C(\sigma(A))$.*

Proof. First, we show $\omega(x^*) = \overline{\omega(x)}$ for all $\omega \in \sigma(A)$. (I.e. $\hat{x^*} = \bar{\hat{x}}$.) For all $x \in A$, one can write $x = x_1 + ix_2$ where $x_i^* = x_i$. Now suppose $y \in A$ and $y^* = y$ and consider $\exp(ity)$.

First supppose $x^* = x$. We know $||\hat{x}||_\infty = r(x)$ so to show $||\hat{x}||_\infty = ||x||$ we only have to show $||x^2|| = ||x||^2$.

Now apply Stone-Weierstrass.

Corollary. *If $x \in A$ is self-adjoint, then $\sigma(x) \subset \mathbf{R}$.*

2. Spectral Permanence Re-emerges

Theorem. *Suppose $B \subseteq A$ are unital C^*-algebras. For all $x \in B$, $\sigma_A(x) = \sigma_B(x)$.*

Proof. Enough to show $\supseteq$. First suppose $x = x^*$. Claim: If $x \in A^{-1}$ then $x \in B^{-1}$. Also, if y^*y is invertible $y^{-1} = (y^*y)^{-1}y^*$.

Corollary. *For all $T \in \mathcal{B}(H)$,*

$$\sigma(T) = \sigma_{C^*(T)}(T).$$

3. Functional Calculus for Normal Operators.

An operator $a \in A$ is said to be **normal** if

$$aa^* = a^*a.$$

Let $X = \sigma_A(a)$.

What we want to do is to be able to "evaluate" functions continuous f on X at a so that $f(a) \in C^*(a)$ and,

(i) $f(a) = c$ if f is the constant function on X, $x \mapsto c$.

(ii) $z(a) = a$

(iii) $\bar{f}(a) = f(a)^*$.

I.e., we want to define a $*$-homomorphism from $C(X)$ to $C^*(a)$.

Proposition. *The map $\omega \in \sigma(C^*(a)) \mapsto \omega(a)$ is a homeomorphism onto X.*

Now we know we have a $*$-isometry $C^*(a) \to C(\sigma(C^*(a)) \cong C(X)$. The inverse is the map we want.

Spectral Mapping Theorem. *Suppose $f \in C(\sigma(a))$, then*

$$\sigma(f(a)) = f(\sigma(a)).$$

Homework problems due Friday, Oct. 10

Let X and Y be compact Hausdorff spaces and let $\theta: C(X) \to C(Y)$ be a homomorphism of complex algebras (do not assume continuity).

1. Suppose θ is an isomorphism. (i) Let $p \in Y$. Show there $\exists !\ q \in X$ such that,

$$\theta f(p) = f(q), \quad \forall f \in C(X).$$

(ii) Show thwere exists a homeomorphism $\phi: Y \to X$ such that,

$$\theta f = f \circ \phi, \quad \forall f \in C(X).$$

(iii) Conclude that θ is a self-adjoint linear map satisfying,

$$||\theta f|| = ||f||, \quad \forall f \in C(X).$$

2. Formulate and prove a theorem that characterizes unital homomorphisms $\theta: C(X) \to C(Y)$.

3. Show that neither $l^1(\mathbf{Z})$ nor the ring of convergent power series on $B[0,1]$ (with the involutions and norms we have defined) are C^*-algebras.

Lecture 18

1. Functional Calculus for Normal Operators. (Conway VIII §2)

An operator $a \in A$ is said to be **normal** if

$$aa^* = a^*a.$$

Let $X = \sigma_A(a)$.

What we want to do is to be able to "evaluate" functions continuous f on X at a so that $f(a) \in C^*(a)$ and,

(i) $f(a) = c$ if f is the constant function on X, $x \mapsto c$.

(ii) $z(a) = a$

(iii) $\bar{f}(a) = f(a)^*$.

I.e., we want to define a $*$-homomorphism from $C(X)$ to $C^*(a)$.

Proposition. *The map $\omega \in \sigma(C^*(a)) \mapsto \omega(a)$ is a homeomorphism onto X.*

Now we know we have a $*$-isometry $C^*(a) \to C(\sigma(C^*(a)) \cong C(X)$. The inverse is the map we want.

Spectral Mapping Theorem. *Suppose $f \in C(\sigma(a))$, then*

$$\sigma(f(a)) = f(\sigma(a)).$$

2. Positive elements of a C^*-algebra (VIII, §3)

Suppose A is a C^*-algebra. Let $\mathrm{Re}(A)$ denote the selfadjoint elements of A.

Definition. *An element a of A is said to be positive, written $a \geq 0$, if $a \in \mathrm{Re}(A)$ and $\sigma(a) \subseteq [0, \infty)$.*

In particular, if $a \in \mathrm{Re}(A)$, $a^2 \geq 0$. In fact, all positive elements are of this form. Also, if $A = C(X)$, X compact, f is positive if and only if $f(x) \geq 0$ for all $x \in X$.

Proposition. *If $a \in \mathrm{Re}(A)$ there are positive unique elements u, v such that $a = u - v$ and $uv = vu = 0$. If a is positive, and n is a positive integer, there exists a unique positive $b \in A$ such that $a = b^n$.*

Proof. Let $M(t) = \max\{t, 0\}$ and $m(t) = -\min\{t, 0\}$.

3. What do we know about spectra?

Suppose T is a continuous operator on a Banach space X.

Definition. *Suppose $\lambda \in \sigma(T)$. Then there are three possibilities.*
(i) λ is an eigenvalue of T. Call these $\sigma_p(T)$, the **point spectrum** *of T.*
(ii) $\lambda - T$ is injective but not surjective but its image is dense. Call this$\sigma_c(T)$, the **continuous spectrum** *of T.*
(iii) $\lambda - T$ is injective but its image is not dense. Call this $\sigma_r(T)$, the **residual spectrum** *of T.*

Suppose A is a C^*-algebra. If $T \in A$ is self-adjoint $\sigma(T) \subseteq \mathbf{R}$ and if T is **unitary** $\sigma(T) \subseteq S^1$. More generally, the spectral radius of any $T \in A$ is $||T||$.

End of proof of:

Proposition. *If $a \in \ Re(A)$ there are positive unique elements u, v such that $a = u - v$ and $uv = vu = 0$.*

1. Adjoining a unit (VIII §1)

Suppose A is a Banach algebra. Define,

$$A^e = \{(a, \lambda) \colon a \in A \lambda \in \mathbf{C}\} =: A \dot{+} \mathbf{C}.$$

Addition is componentwise and

$$(a, \lambda)(b, \mu) = (ab + \lambda b + \mu a, \lambda\mu)$$

so $(0, 1)$ is the identity. Moreover,

$$||(a, \lambda)|| = ||a|| + |\lambda|$$

Notes: (i) $||1|| = 1$.

(ii) If A is a involutive Banach algebra, we can make A^e one too.

(iii) Now we can set $\sigma(a) = \sigma_{A^e}(a)$.

(iv) $0 \in \sigma(a)$ $\forall a \in A$.

(v) What happens when A has a unit?

2. Some facts about non-unital Banach algebras

Suppose A is an Abelian Banach algebra without an identity. Let $\sigma(A)$ be the set of homomorphisms of A onto $\mathbf{C}$. Every element in $\sigma(A)$ is continuous. The set $\sigma(A)$ given the weak* topology as a subset of A' with is locally compact. The Gelfand transform

$$a \mapsto \hat{a} : \omega \mapsto \omega(a)$$

is a continuous map of A into $C_0(\sigma(A))$. If A is a C^*-algebra, the Gelfand transform is $*$-isometry of A onto $C_0(\sigma(A))$. Is $\sigma(A^e)$ the one point compactication of $\sigma(A)$?

3. Representations

Suppose A is a not necessarily unital involutive Banach algebra such that $||a^*|| = ||a||$. Such an algebra is called a $*$-Banach algebra. (Eg., $l^1(\mathbf{Z})$.)

Definition. *A* **representation** *of A is a $*$-homomorphism π of A into $\mathcal{B}(H)$ for some Hilbert space H. Denote the set of these $rep(A, H)$.*

Note, we do not assume π is bounded.

Definition. *Set $N_\pi = \{x \in H \colon \pi(A)x = 0\}$. The representation π is said to be* **non-degenerate** *if $N_\pi = 0$, i.e., if*

$$\pi(A)x = \{0\} \Rightarrow x = 0.$$

Remarks: (i) $N_\pi = (\pi(A)H)^\perp$

(ii) π is non-degenerate if and only if

$$H = H_\pi := \overline{span}\{\pi(A)H\}$$

(iii) $H = H_\pi + N_\pi$.

Thus we can always suppose π is non-degenerate by replacing π with its projection π_e to H_π.

(iv) $||\pi(a)|| = ||\pi_e(a)||$.

(v) If $1 \in A$, π non-degenerate implies $\pi(1) = 1$.

Theorem. *If $\pi \in rep(A, H)$, $||\pi|| \leq 1$.*

Lecture 20

1. Representations are Continuous

Theorem. *Suppose A is a $*$-Banach algebra and H is Hilbert space. If $\pi \in rep(A, H)$, $||\pi|| \leq 1$.*

Proof. WOLG π is non-degenerate.

Case 1: $1 \in A$ so $\pi(1) = 1$. Claim: If $a \in A$, $\sigma(\pi(a)) \subseteq \sigma(a)$.

Case 2: $1 \notin A$. Extend π to A^e, via

$$\pi((a, \lambda)) = \pi(a) + \lambda.$$

2. Examples of representations

(i) Suppose H is a Hilbert space and N is a normal operator in $\mathcal{B}(H)$. Let $X = \sigma(N)$. Then we have an element in $rep(C(X), H)$.

(ii) If X is a compact Hausdorff space and μ is a Borel measure on X, $\pi(f) = M_f$ is an element of $rep(C(X), L^2(X, \mu))$.

(iii) We have a representation of $l^1(\mathbf{Z})$ into $\mathcal{B}(L^2[0, 1])$.

3. The Spectral Theorem: Foreshadowing

Theorem. *Suppose H is a finite dimensional Hilbert space. Let N be a normal operator on H, let $\lambda_1, \ldots, \lambda_n$ be the distinct eigenvalues of N and let E_i be the orthogonal projection of H onto $Ker(N - \lambda_i)$. Then,*

$$N = \sum_{i=1}^{n} \lambda_i E_i.$$

This follows from the functional calculus for finite dimernsional Hilbert spaces. We want to generalize this to arbitrary Hilbert spaces.

Definition. *Suppose X is a set and Ω is a σ-algebra on X and H is a Hilbert space. A* **spectral measure** *on (X, Ω) is a function $E: \Omega \to \mathcal{B}(H)$ such that*

(a) For each $S \in \Omega$, $E(S)$ is a projection.

(b) $P(\emptyset) = 0$ and $P(X) = 1$.

(c) $E(S \cap T) = E(S)E(T)$ for $S, T \in \Omega$.

(d) If $\{S_n\}_{n=0}^{\infty}$ is a disjoint sequence of elements of Ω, then

$$E(\bigcup_{n=0}^{\infty} S_n) = \sum_{n=0}^{\infty} E(S_n).$$

Remark: In fact, (a), (b) and (c) imply the series of operators above converges in $\mathcal{B}(H)$ (exercise).

If X is a compact Hausdorff space, $f \in C(X) \mapsto \int f dE$ is a representation. Now we can state:

The Spectral Theorem. *Suppose N is a normal operator on a Hilbert space H. Then there is a unique spectral measure on the Borel σ-algebra of $\sigma(N)$, E_N,*

$$N = \int z dE_N(z),$$

if U is a non-empty open subset of $\sigma(N)$, $E_N(U) \neq 0$ and if $A \in \mathcal{B}(H)$, $AN = NA$ and $AN^* = N^*A$ if and only if $AE_N(S) = E_N(S)A$ for every Borel set S.

Homework problems due October 18

1. Suppose X is a Banach space. (i) Show that if I is a non-zero closed ideal which is unital as a Banach algebra then $I = \mathcal{B}(X)$. (ii) Conclude that if X is infinite dimensional, the algebra of compact operators from X to itself is not unital. (iii) Show the set of nuclear operators N (i.e., operators in the image of $X' \hat{\otimes} X$) from X to itself is a closed ideal and conclude N is not unital as an algebra if X is infinite dimensional.

2. A representation $\pi \in rep(A, H)$ is said to be **cyclic** if there is a vector $e \in H$ such that $\overline{\pi(A)e} = H$. Show example (ii) above is cyclic.

3. Prove that properties (a)-(c) of the spectral measure imply the sum in (d) converges.

Lecture 21

Exercise: Self-adjoint projectors are orthogonal projectors. Also, Multiplication operators are normal.

1. The extended functional calculus.

Suppose $N \in \mathcal{B}(H)$ is a normal operator. We want to define a spectral measure on $X = \sigma(T)$. This turns out to be equivalent to an extention of our representation $\pi\colon C(X) \to \mathcal{B}(H)$ to a representation $\tilde{\pi}\colon B(X) \to \mathcal{B}(X)$ where $B(X)$ the set of bounded Borel measurable functions on X with a certain continuity property. Recall:

Definition. *A real valued* **Borel measurable** *function on a locally compact set X is a function f such that $f^{-1}((a,b))$ is a Borel set for all real $a < b$. A complex valued function is Borel measurable if and only if its real and imaginary parts are.*

General setting

Suppose X is a compact Hausdorff space, $C(X)$ the continuous functions on X and $B(X)$ the set of bounded Borel measurable functions on X.

Note: 1. $B(X)$ is a unital C^* algebra.

$$||f|| = \sup_{x \in X} |f(x)|.$$

2. $B(X)$ is inseparable if X is uncountable.

3. If $f_n \in B(X)$ is a sequence of functions satisfying

(i) $||f_n|| \leq M < \infty$

(ii) $\lim_{n\to\infty} = f(x)$ exists for all $x \in X$.

Then $f \in B(X)$.

4. Let $f \in B(X)$, $\epsilon > 0$ then $\exists$ a simple function f_0 such that $||f - f_0|| < \epsilon$.

Definition. *A representation $\pi\colon B(X) \to \mathcal{B}(H)$ is said to be* **admissable** *if for all sequences $f_1, \ldots, f_n, \ldots$ such that $||f_n|| \leq M < \infty$ and $\lim_n f_n(x) = 0$, $\forall x \in X$,* *one has*

$$\lim_n ||\pi(f_n)h|| = 0, \ \forall h \in H.$$

2. Topologies on $\mathcal{B}(H)$.

1. Norm topology.

2. Strong and σ-strong:

(a) Suppose $S = \{h_1, \ldots, h_n\} \subset H$, then we define the strong semi-norm;

$$|T|_S = (\textstyle\sum_{k=1}^{n} ||Th_k||^2)^{1/2}.$$

(b) Suppose $S = \{h_1, h_2, \ldots\} \subseteq H$ such that $\sum_{k=1}^{\infty} < \infty$, then we define the σ-strong semi-norm; $|T|_S = (\sum_{k=1}^{\infty} ||Th_k||^2)^{1/2}$.

3. Weak and σ-weak:

(a) Weak seminorms; $|T| = |\sum_{j=1}^{n} \langle Th_i, g_i \rangle|$.

(b) σ-weak semi-norms; $|T| = |\sum_{i=1}^{\infty} \langle Th_i, g_i \rangle|$, where $\sum ||h_i||^2 < \infty$ and $\sum ||g_i||^2 < \infty$. $\blacksquare$

4. Basic neighbnorhoods of 0:

(a) strong; $\{T\colon ||Th_1|| < 1, \ldots, ||Th_n|| < 1\}$.

(b) σ-strong $\{T\colon \sum_{i=1}^{\infty} ||Th_i||^2 < 1, \ \text{if} \sum_i ||h_i||^2 < \infty\}$.

(c) weak; $\{T\colon |\langle Th_i, g_i \rangle| < 1, 1 \leq i \leq n\}$.

(d) σ-weak; $\{T\colon |f_j(T)| < 1, 1 \leq j \leq m\}$ where $f_j(T) = \sum_{i=1}^{\infty} \langle Th_i^j, g_i^j \rangle$, etc.

Remark: Let $\pi\colon B(X) \to \mathcal{B}(H)$ be a representation. TFAE:

(i) π is admissable.

(ii) For all bounded sequences $f_n \in B(X)$ such that $f_n(p) \to f(p)$, $\forall p \in X$, $\pi(f_n) \to \pi(f)$ strongly.

(iii) $\ldots$, $\pi(f_n) \to \pi(f)$ weakly.

Proof. Let $g_n = f_n - f$. For (iii) $\Rightarrow$ (ii), consider $||\pi(g_n)h||^2$.

Mala pointed out that we don't need the ideal in Problem 1 to be closed and I don't know if the ideal of nuclear operators is closed.

1. Some Basic Neighbnorhoods of 0:

Suppose $h_1, \ldots, h_n, g_1, \ldots, g_n \in H$.
Strong; $\{T\colon ||Th_1|| < 1, \ldots, ||Th_n|| < 1\}$.
Weak; $\{T\colon |\langle Th_1, g_1\rangle| < 1, \ldots, |\langle Th_n, g_n\rangle| < 1\}$.
Recall: A representation $\pi\colon B(X) \to \mathcal{B}(H)$ is said to be admissible if for all sequences $f_1, \ldots, f_n, \ldots$ such that $||f_n|| \le M < \infty$ and $\lim_n f_n(p) = 0,\ \forall p \in X$, one has

$$\lim_n ||\pi(f_n)h|| = 0,\ \forall h \in H.$$

Lemma. *Let $\pi\colon B(X) \to \mathcal{B}(H)$ be a representation. Then, the following are equivalent:*
(i) π is admissible.
(ii) For all bounded sequences $f_n \in B(X)$ such that $f_n(p) \to f(p),\ \forall p \in X$, $\pi(f_n) \to \pi(f)$ strongly.
(iii) For all bounded sequences $f_n \in B(X)$ such that $f_n(p) \to f(p),\ \forall p \in X$, $\pi(f_n) \to \pi(f)$ weakly.

Proof. Replace f_n by $g_n = f_n - f$. For (iii) $\Rightarrow$ (ii), consider $||\pi(g_n)h||^2$.

2. A General Extension Theorem

Theorem. *Let $\pi \in rep(C(X), H)$ be a non-degenerate representation. Then there is a unique admissible representation $\tilde{\pi} \in rep(B(X), H)$ such that*

$$\tilde{\pi}|_{C(X)} = \pi.$$

First we prove uniqueness.
Suppose $\sigma \in rep(B(X), H)$ is admissable. Define for all $x, y \in H,\ f \in B(X)$

$$\mu(\sigma)_{x,y}(f) = \langle \sigma(f)x, y\rangle.$$

Then:

1. $\mu_{x,y}$ is a finite Borel measure.

Proof.

(i) Countable Additivity. Suppose E is a disjoint union of the Borel sets $E_1, \ldots, E_n, \ldots$ and

$$f_n = \chi_{\cup_{i=1}^n E_i} = \sum_{i=1}^n \chi_{E_i}.$$

(ii) $\int f d\mu_{x,y} = \langle \sigma(f)x, y \rangle$

Now suppose σ and τ are two admissible extensions of π. Use Riesz-Marcov.

To prove existence, we have to talk about spectral measures.

3. Spectral Measures and admissible representations

Recall: Suppose (X, Ω) is a Borel space and H is a Hilbert space. A spectral measure on (X, Ω) is a function $E \colon \Omega \to \mathcal{B}(H)$ such that

(a) For each $S \in \Omega$, $E(S)$ is a (orthogonal) projection.

(b) $E(\emptyset) = 0$ and $E(X) = 1$.

(c) $E(S \cap T) = E(S)E(T)$ for $S, T \in \Omega$.

(d) If $\{S_n\}_{n=0}^\infty$ is a disjoint sequence of elements of Ω, then

$$E(\bigcup_{n=0}^\infty S_n) = \sum_{n=0}^\infty E(S_n).$$

Proposition. *Suppose $\pi \in rep(B(X), H)$ is admissible. Then $P(S) = \pi(\chi_S)$ is s spectral measure.*

Theorem. *There is a one-to-one correspondence between spectral messures on X and admissible representationsn of $B(X)$.*

Lecture 23

1. Back to the General Extension Theorem

Theorem. *Let $\pi \in rep(C(X), H)$ be a non-degenerate representation. Then there is a unique admissible representation $\tilde{\pi} \in rep(B(X), H)$ such that*

$$\tilde{\pi}|_{C(X)} = \pi.$$

End of the proof of uniqueness:

Suppose $\sigma \in rep(B(X), H)$ is admissable. Define for all $x, y \in H$, $f \in B(X)$

$$\mu_{x,y}(\sigma)(f) = \langle \sigma(f)x, y \rangle.$$

We showed: $\mu_{x,y}(\sigma)$ is a finite Borel measure.

Now suppose σ and τ are two admissible extensions of π. Then, by Riesz-Marcov, $\mu_{x,x}(\sigma) = \mu_{x,x}(\tau)$ for all x.

2. Existence.

Lemma. *For all $x, y \in H$ $\exists \mu_{x,y} \in M(X)$ (finite Baire measures on X) such that $\forall f \in C(X)$,*

$$\int f d\mu_{x,y} = \langle \pi(f)x, y \rangle.$$

Claim: $||\mu_{x,y}|| \leq ||x|| \cdot ||y||$.

Finally, claim: for any $f \in B(X)$, $\exists!$ $\tilde{\pi}(f) \in \mathcal{B}(X)$ such that

$$\langle \tilde{\pi}(f)x, y \rangle = \int f d\mu_{x,y}.$$

Now we have to show $\tilde{\pi}$ is an admissable representation. I.e.,

(A) $\tilde{\pi}(\bar{f}) = \tilde{\pi}(f)^*$.

(B) $\tilde{\pi}(fg) = \tilde{\pi}(f)\tilde{\pi}(g)$.

(C) $\tilde{\pi}$ is admissable.

(A) Claim: $\overline{\mu}_{x,y} = \mu_{y,x}$.

Now

$$\langle \tilde{\pi}(f)^* x, y \rangle =$$

(B) Claim: $\forall f \in B(X)$, $f\mu_{x,y} = \mu_{\pi(f)x,y} = \mu_{x,\pi(f)^*y}$. First suppose $f \in C(X)$. Suppose $g \in C(X)$. Consider,

$$\int g f d\mu_{x,y}$$

(C) Suppose $f_n \in B(X)$, $||f_n|| \leq M < \infty$, $f_n \to 0$ pointwise. (Use the dominated convergence theorem.)

Homework problems due Fri. Oct. 25

Let $H = L^2[0,1]$ and $Af(x) = xf(x)$ for $f \in H$, $x \in [0,1]$.

1. Show there is a unique representation $\pi \in rep(C[0,1], H)$ satisfying $\pi(1) = 1$ and $\pi(u) = A$ where $u(x) = x$. Give an explicit description of this representation.

2. Compute, explicity, the admissible extension $\tilde{\pi}$ of π to $B[0,1]$.

3. Let E be the spectral measure attached to A. Show that E is supported on $[0,1]$ and show that for $S \in B([0,1])$, $E(S) = 0$ if and only if $m(S) = 0$ where m is the Lebesgue measure on $[0,1]$.

4. Let $H = l^2(\mathbf{Z})$. Let U be the unitary operator on H which takes $e_n := \chi_{\{n\}}$ to e_{n+1}. (i) Show $\sigma(U) = S^1$. (ii) Compute the spectral measure attached to U.

1. Some clarfication:

In the existence part of the proof of,

Theorem. *Let $\pi \in rep(C(X), H)$ be a non-degenerate representation. Then there is a unique admissible representation $\tilde{\pi} \in rep(B(X), H)$ such that*

$$\tilde{\pi}|_{C(X)} = \pi.$$

we showed: For all $x, y \in H$ $\exists! \mu_{x,y} \in M(X)$ such that $\forall f \in C(X)$,

$$\int f d\mu_{x,y} = \langle \pi(f)x, y \rangle$$

and $||\mu_{x,y}|| \leq ||x|| \cdot ||y||$. I claimed that it followed that for any $f \in B(X)$, $\exists!$ $\tilde{\pi}(f) \in \mathcal{B}(X)$ such that

$$\langle \tilde{\pi}(f)x, y \rangle = \int f d\mu_{x,y}.$$

For this we need, $\mu_{x,y} = \overline{\mu_{y,x}}$ and a corollary of the Riesz Representation Theorem.

2. Spectral Measures and Admissible Representations

Theorem. *There is a one-to-one correspondence between spectral measures on X and admissible representations of $B(X)$.*

Recall: Suppose (X, Ω) is a Borel space and H is a Hilbert space. A spectral measure on (X, Ω) is a function $E \colon \Omega \to \mathcal{B}(H)$ such that

(a) For each $S \in \Omega$, $E(S)$ is a (orthogonal) projection.

(b) $E(\emptyset) = 0$ and $E(X) = 1$.

(c) $E(S \cap T) = E(S)E(T)$ for $S, T \in \Omega$.

(d) If $\{S_n\}_{n=0}^{\infty}$ is a disjoint sequence of elements of Ω, then

$$E(\bigcup_{n=0}^{\infty} S_n) = \sum_{n=0}^{\infty} E(S_n).$$

Proof of theorem.

First suppose σ is an admissable representation. Set $E(S) = \sigma(\chi_S)$.

Lemma. *For each a spectral measure, P, with values in $\mathcal{B}(H)$ there exists a unique bounded linear map $\sigma_P\colon B(X) \to \mathcal{B}(H)$ such that $||\sigma_P(f)|| \le ||f||$*

$$\sigma_P(\chi_E) = P(E).$$

Proof. Uniqueness is easy. For existence, define, for $S \in \Omega$,

$$\mu_{x,y}(S) = \langle P(S)x, y\rangle.$$

Claim: The map $(x, y) \mapsto \mu_{x,y}$ is a sesquelinear map of $H \times H$ into $M(X)$ such that

$$||\mu_{x,y}|| \le ||x|| \cdot ||y||,$$

where the left hand side is the total variation of the measure. Let $S_1, \ldots, S_n$ be disjoint Borel subsets of X. Then, $\exists \alpha_i$, $|\alpha_i| = 1$, such that

$$\sum_i |\mu_{x,y}(S_i)| = \sum_i \alpha_i \langle P(S_i)x, y\rangle$$

By Riesz, $\forall f \in B(X)$, $\exists! \sigma_P(f) \in \mathcal{B}(H)$ such that

$$\langle \sigma_P(f)x, y\rangle = \int f d\mu_{x,y} \ , \quad \forall x, y \in H.$$

Moreover, $||\sigma_P(f)|| \le ||f||$. Claim: σ_P is an admissable representation.

3. The Spectral Theorem

The Spectral Theorem. *Suppose N is a normal operator on a Hilbert space H. Then there is a unique spectral measure on the Borel σ-algebra of $\sigma(N)$, E_N,*

$$N = \int z dE_N(z).$$

Moreover, (i) if U is a non-empty open subset of $\sigma(N)$, $E_N(U) \neq 0$ and (ii) if $A \in \mathcal{B}(H)$, $AN = NA$ and $AN^* = N^*A$ if and only if $AE_N(S) = E_N(S)A$ for every Borel set S .

Lecture 25

1. Fraser's improvement

Recall, we stated,

Theorem. *Let $\pi \in rep(C(X), H)$ be a non-degenerate representation. Then there is a unique admissible representation $\tilde{\pi} \in rep(B(X), H)$ such that*

$$\tilde{\pi}|_{C(X)} = \pi.$$

But we never needed π to be non-degenerate.
Also, we proved that if P is the spectral measure corresponding to an admissable representation σ of $B(X)$ into $\mathcal{B}(H)$,

$$\sigma(f) = \int f dP.$$

2. The end of the proof of the Spectral Theorem

The Spectral Theorem. *Suppose N is a normal operator on a Hilbert space H. Then there is a unique spectral measure on the Borel σ-algebra of $\sigma(N)$, E_N,*

$$N = \int z dE_N.$$

Moreover, (i) if U is a non-empty open subset of $\sigma(N)$, $E_N(U) \neq 0$ and (ii) if $A \in \mathcal{B}(H)$, $AN = NA$ and $AN^* = N^*A$ if and only if $AE_N(S) = E_N(S)A$ for every Borel set S.

Proof. Let $X = \sigma(X)$ and for $h \in B(X)$, let $h(N) = \int h dE_N$ denote its image in $\mathcal{B}(H)$ under the extended functional calculus.
For (ii), we first prove that if A commutes with N and N^* then A commutes with $f(N)$ for all f in $C(X)$.
Now, let $u_n \in C(X)^+$, $u_n \uparrow \chi_S$ and consider $\langle A\chi_S(N)g, h\rangle$.

3. Compact operators

Recall, if X be a Banach space, a linear map L from X to X is said to be **compact** if the set $\{L(x) : |x| < 1\}$ has compact closure in X. Let $\mathcal{B}_0(X)$ be the set of compact operators.

Proposition. *Set $\Delta_\epsilon = \{z \in \sigma(N) \colon |z| > \epsilon\}$. Then, N is compact if and only if for every $\epsilon > 0$, $\chi_{\Delta_\epsilon}(N)$ has finite rank.*

Corollary. *Every compact operator on a Hilbert space is a limit of operators of finite rank.*

Corollary. *If N is a normal compact operator on a Hilbert space and $\epsilon > 0$, then Δ_ϵ is finite.*

1. The end of the end of the proof of the Spectral Theorem

Theorem. *Suppose N is a normal operator on a Hilbert space H and E_N is the corresponding spectral measure. Then, if $A \in \mathcal{B}(H)$, $AN = NA$ and $AN^* = N^*A$ if and only if $AE_N(S) = E_N(S)A$ for every Borel set S.*

Proof. Let $X = \sigma(N)$ and for $h \in B(X)$, let $h(N) = \int h dE_N$. We know this is the unique admissible extention of the functional calculus. Recall, for $x, y \in H$ we produced a Borel measure $\mu_{x,y}$ such that for all $f \in B(X)$,

$$\inf f d\mu_{x,y} = \langle f(N)x, y \rangle.$$

We proved that if A commutes with N and N^* then A commutes with $f(N)$ for all f in $C(X)$. Now, let S be an open set in X and $u_n \in C(X)^+$, $u_n \uparrow \chi_S$. and consider $\langle A\chi_S(N)g, h \rangle$.
Now let

$$\Omega = \{S\colon S \text{ is a Borel set in } X \text{ and} A\chi_S(N) = \chi_S(N)A\}.$$

2. Compact operators on a Hilbert space

We keep the notation of the last section. We proved,

Proposition. *Set $\Delta_\epsilon = \{z \in \sigma(N)\colon |z| > \epsilon\}$. Then, N is A NORMAL compact if and only if for every $\epsilon > 0$, $\chi_{\Delta_\epsilon}(N)$ has finite rank.*

Corollary. *Every compact operator on a Hilbert space is a limit of operators of finite rank.*

Corollary. *If N is a normal compact operator on a Hilbert space and $\epsilon > 0$, then Δ_ϵ is finite.*

Theorem. *If H is a separable Hilbert space, then the only ideal in $\mathcal{B}(H)$ which contains a non-compact operator is $\mathcal{B}(H)$.*

We will need,

Proposition. *Suppose A is a C^* algebra. If $a \in A$ then $a^*a \geq 0$ and a lies in the ideal generated by a^*a.*

Proof of theorem. Note: All separable Hilbert spaces are isomorphic.

Corollary. *If H is a separable Hilbert space, the only closed ideals in $\mathcal{B}(H)$ are $\{0\}$, $\mathcal{B}_0(H)$ and $\mathcal{B}(H)$.*

For this we need,

Proposition. *If H is a Hilbert space, every non-zero ideal in $\mathcal{B}(H)$ contains all operators of finite rank.*

Homework problems due Fri. Nov. 1

Consider the Hilbert spaces $H_1 = l^2(\mathbf{Z}^+)$, $\quad H_2 = L^2[0,1]$, $\quad H_3 = L^2([0,1] \times [0,1])$. Let $\{r_1, r_2, \ldots\}$ be a countable dense subset of $[0,1]$. Define operators A_i on H_i by, $A_1 f(n) = r_n f(n)$, $\quad A_2 f(x) = x f(x)$ and $A_3 f(x,y) = x f(x,y)$.

1. Show that if H is a Hilbert space and $A \in \mathcal{B}(H)$, then $A^* = A$ if and only if $\langle Ax, x \rangle$ if real for all $x \in H$.
2. Show that $\sigma(A_1) = \sigma(A_3) = [0,1]$.
3. Compute the spectral measures E_{A_1} E_{A_2} and E_{A_3} of A_1, A_2 and A_3. (You may cite previous problem sets for A_2.)
4. Show there is a Baire set $X \subseteq [0,1]$ such that E_{A_1} is concentated on X and E_{A_2} is conentated on $[0,1] \backslash X$.

Lecture 27

1. Spectral Theorem for Compact Normal Operators

Theorem. *Suppose N is a compact normal operator on the Hilbert space H. Then the spectrum of N is countable and if S is the set of non-zero elements in $\sigma(N)$ and P_λ is the orthogonal projection onto the eigenspace of N with eigenvalue λ,*

$$N = \sum_{\lambda \in S} \lambda P_\lambda.$$

Lemma. *If N is a normal bounded operator on a Hilbert space H, $h \in H$ and $\lambda \in \mathbf{C}$, then if $Nh = \lambda h$, $N^* h = \bar{\lambda} h$.*

Proposition. *If N is a normal bounded operator on a Hilbert space H and $h \in H$, $\lambda \in \mathbf{C}$ such that $Nh = \lambda h$, then $f(N)h = f(\lambda)h$ if f is a Borel function on $\sigma(N)$ such that either f is continuous or f vanishes in a neighborhood of λ.*

Proof. (Exercise) Is the converse true for continuous functions? What is the right statement for Borel functions which includes the statement for continuous functions?

Proposition. *Suppose N is a normal operator and λ is an isolated point of the spectrum of N. Then if $U = \{\lambda\}$, $\chi_U(N)$ is the orthogonal projector onto the λ-eigenspace of N.*

Proof. First $z\chi_U(z) = \lambda\chi_U(z)$. Next if $Nx = \lambda x$,
Proof of theorem.

2. The Riesz projector

(For me projectors are not necessarily orthogonal.)
(See Gohberg-Krejn, Introduction to the Theory of Linear Nonselfadjoint Operators. (1969).)

For a bounded operator L on a Banach space, let $\rho(L) = \mathbf{C} \backslash \sigma(L)$. This is called the **resolvant set** of L.

Suppose X is a Banach space and A is a bounded linear operator on X. For $\lambda \in \rho(A)$ the operator $R(\lambda) := R_A(\lambda) = (A - \lambda)^{-1}$ is called a **resolvant**. The function $\lambda \mapsto R(\lambda)$ from $\mathbf{C} \backslash \sigma(A)$ to $\mathcal{B}(X)$ is an analytic function (exercise).

Suppose G is a bounded open subset of $\mathbf{C}$ whose boundary, $\Gamma := \Gamma_G$ is a finite union rectifiable closed curves contained in $\rho(A)$. Such an open set is called an **admissable domain**. Orient Γ_G positively with respect to G. Let

$$P_\Gamma := P_\Gamma(A) = -\frac{1}{2\pi i} \int_\Gamma R(\lambda) d\lambda.$$

Proposition. *The operator $P_\Gamma(A)$ is a projector which commutes with A such that the complementary subspaces of X,*

$$L_\Gamma = P_\Gamma X \quad \text{and} \quad (1 - P_\Gamma)X,$$

are closed and invariant under the action of A. Moreover, (i) the spectrum of the restriction of A to L_Γ is contained in G and (ii) the spectrum of its restriction to N_Γ is contained in $\overline{\sigma(A) \backslash G}$ (?).

(See Riesz-Nagy, Functional Analysis 1955, §148.)

Proof. Let $S = G \cap \sigma(A)$. First if G' is another admissable domain such that $G \cap \sigma(A) = S$ with boundary Γ' then $P_\Gamma(A) = P_{\Gamma'}(A)$. Call this common value P_S. Now suppose, $S \subset G_{\Gamma'} \subset G_\Gamma$. Then,

$$P_S^2 = (-\frac{1}{2\pi i})^2 \int_\Gamma \int_{\Gamma'} R_\lambda R_{\lambda'} d\lambda d\lambda'$$

Next claim:

$$(A - \lambda) \frac{1}{2\pi i} \int_\Gamma \frac{R_z}{z - \lambda} dz = \begin{cases} -P_S & \lambda \in \mathbf{C} \backslash \bar{G} \\ 1 - P_S & \lambda \in G \end{cases}.$$

1. The Riesz projector

Suppose X is a Banach space and A is a bounded linear operator on X. Suppose G is an admissable domain for A with boundary $\Gamma := \Gamma_G$. an **admissable domain**. Let $S = G \cap \sigma(A)$ and $S' = \sigma(A) \backslash S$. Orient Γ_G positively with respect to G. Recall, we defined

$$P_S = -\frac{1}{2\pi i}\int_\Gamma R(\lambda)d\lambda.$$

and we proved it is a projector and depends only on S. Also, if

$$L_S = P_S X \quad \text{and} \quad N_S = (1 - P_S)X$$

then L_S and N_S are closed subspaces invariant under the action of S. We used the fact that

$$\frac{1}{2\pi i}\int_\Gamma \frac{1}{z-\lambda}dz = \begin{cases} 1 & \text{if } \lambda \in G \\ 0 & \text{if } \lambda \notin \bar{G} \end{cases}$$

to prove,

$$(A-\lambda)\frac{1}{2\pi i}\int_\Gamma \frac{R_z}{z-\lambda}dz = \begin{cases} -P_S & \lambda \in \mathbf{C}\backslash\bar{G} \\ 1-P_S & \lambda \in G \end{cases}.$$

Proposition. *(i) the spectrum of the restriction of A to L_Γ is S,*
(ii) the spectrum of its restriction to N_Γ is S' and
(iii) $P_{S'} = 1 - P_S$.

Proof. Indeed, $\rho(A|_{L_S}) \supseteq \rho(A)$ and $\rho(A|_{N_S}) \supseteq \rho(A)$.

2. Radical Multiplicty

Suppose A is a bounded operator on a Banach space X. A point $\lambda \in \mathbf{C}$ is called **normal** for A if there exists an integer $n \geq 0$ such that

$$N_\lambda := (A-\lambda)^n X = (A-\lambda)^{n+1} X, \quad L_\lambda := Ker(A-\lambda)^n = Ker(A-\lambda)^{n+1}$$

L_λ is closed and N_λ is finite dimensional. It follows that $A - \lambda$ is invertible on N_λ and

$$X = N_\lambda \oplus L_\lambda.$$

The subspace L_λ is called the **radical** subspace pf A relative to λ. I call the elements of L_λ **generalized eigenvectors**. Any point in $\rho(A)$ and any non-zero point in the spectrum of a compact operator is normal, (we proved this for compact normal operators on a Hilbert space). If S is a finite set of normal points for A, we set

$$\nu_S(A) = \sum_{\lambda \in S} \dim(N_\lambda(A)).$$

The theorem we want to prove is:

Theorem. *Suppose G is an admissable domain for A such that $S := G \cap \sigma(A)$ is a finite set of normal points of A. Then there exists a constant $\rho > 0$ depending only on A and G such that for all $B \in \mathcal{B}(X)$ such that $||A - B|| < \rho$, G is an admissable domain for B, $\sigma(B) \cap G =: T$ is a finite set of normal points for B and*

$$\nu_S(A) = \nu_T(B).$$

Theorem. *For a point $\lambda_0 \in \sigma(A)$ to be a normal point for A it is necessary and sufficient the λ_0 be an isolated point of $\sigma(A)$ and P_{λ_0} be a finite rank operator. Moreover, in this case $P_{\lambda_0} X = L_\lambda$. (In other words, our notation is consistent.)*

Proof. Suppose $\lambda_0 \in \sigma(A)$ is normal and let A_1 and A_2 be the restrictions of A to L_{λ_0} and N_{λ_0}. Let $R = (A_2 - \lambda_0)^{-1}$. Then,

$$(A_l - \lambda)^{-1} =$$

$$(A_2 - \lambda)^{-1} =$$

Lecture 29

Suppose A is an operator on a Banach space X.

1. Radical Multiplicity

A general lemma we have used a couple of times is,

Lemma. *Suppose $X = Y \oplus Z$ and Y and Z are A-stable closed subspaces. Then $\sigma(A) = \sigma(A|_Y) \cup \sigma(A|_Z)$. Or equivalently, $A - \lambda$ is invertible if and only if its restrictions to Y and Z are.*

If λ is an isolated point of $\sigma(A)$, let $P_\lambda = P_{\{\lambda\}}$.

Theorem. *For a point $\lambda_0 \in \sigma(A)$ to be a normal point for A it is necessary and sufficient that λ_0 be an isolated point of $\sigma(A)$ and P_{λ_0} be a finite rank operator. Moreover, in this case $P_{\lambda_0}X = L_\lambda$.*

Proof. Suppose $\lambda_0 \in \sigma(A)$ is normal and let A_1 and A_2 be the restrictions of A to L_{λ_0} and N_{λ_0}. Let $n = \dim L_{\lambda_0}$ and P be the projection of X onto L_{λ_0} with kernel N_{λ_0}.
We showed that for $\lambda \neq \lambda_0$ but close to λ_0, $(A - \lambda)^{-1} =$

$$(\lambda_0 - \lambda)^{-1} \sum_{k=0}^{n} \Big(\frac{A_1 - \lambda_0}{\lambda - \lambda_0}\Big)^k P + (A_2 - \lambda_0)^{-1} \sum_{k=0}^{\infty} \Big(\frac{\lambda - \lambda_0}{A_2 - \lambda_0}\Big)^k (1 - P).$$

Theorem. *Suppose G is an admissible domain for A with boundary Γ. Then every point of G is normal if and only if the the the Riesz projector P_Γ has finite rank.*

Proof. Suppose P_Γ has finite rank. Let $L = P_\Gamma X$ and $N = (1 - P_\Gamma)X$. Let A_L and A_N be the corresponding restrictions. We know $\sigma(A_L)$ is finite, disconnected from $\sigma(A_N)$ and equal to $\sigma(A) \cap G$.

For the other direction we need another lemma on the Riesz projector,

Lemma. *Suppose S and T are to disjoint closed-open subsets of $\sigma(A)$. Then,*

$$P_S P_T = P_T P_S = 0.$$

Proof. (Exercise.)

Corollary. *If S is a finite discrete subset of $\sigma(A)$,*

$$P_S = \sum_{\lambda \in S} P_\lambda \quad \text{and} \quad P_\lambda P_{\lambda'} = 0 \;\; \text{if } \lambda \neq \lambda'.$$

Back to the proof of the theorem.

2. Towards Stability of Radical Multiplicity

Lemma. *Suppose P and Q are two projectors in $\mathcal{B}(X)$ such that $||P - Q|| < 1$. Then,* $\dim(PX) = \dim(QX)$ *(which may be infinite).*

Proof. First, claim: If $f \in QX$, $||Pf|| \geq (1 - ||P - Q||)||f||$.

Homework problems due Nov. 8

1. Show if N is a normal bounded operator on a Hilbert space H and $h \in H$, $\lambda \in \mathbf{C}$ such that $Nh = \lambda h$, then $f(N)h = f(\lambda)h$ if f is a Borel function on $\sigma(N)$ such that either f is continuous or f vanishes in a neighborhood of λ. Is the converse true for continuous functions? Formulate a statement for Borel functions which includes the statement for continuous functions?
2. If A is a normal operator on a Hilbert space H and S is closed open subset of $\sigma(A)$ and χ is the chacteristic function of S, show $P_S = \chi(A)$. (Hint: First show that if P and Q are commuting projectors which commute with A, then $\sigma(PQA) \subseteq \sigma(PA) \cap \sigma(QA)$.)

Lecture 30

Suppose A is a bounded operator on a Banach space X.

1. Normal points

We proved,

Theorem. *Suppose G is an admissible domain for A with boundary Γ. Then every point of G is normal if and only if the the the Riesz projector P_Γ has finite rank.*

Corollary. *In this case $P_\Gamma(A)X$ is the direct sum of the the radical subspaces of A corresponding to points in $G \cap \sigma(A)$ so the rank of $P\Gamma(A)$ is the sum of the radical multiplicities of A corresponding to these points.*

2. Stability of Radical Multiplicity

Recall, if S is a finite set of normal points for A,

$$\nu_S(A) = \sum_{\lambda \in S} \dim(N_\lambda(A)).$$

Theorem. *Suppose G is an admissable domain for A such that $S := G \cap \sigma(A)$ is a finite set of normal points of A. Then there exists a constant $\rho > 0$ depending only on A and G such that for all $B \in \mathcal{B}(X)$ such that $||A - B|| < \rho$, G is an admissable domain for B, $\sigma(B) \cap G =: T$ is a finite set of normal points for B and*

$$\nu_S(A) = \nu_T(B).$$

In fact, if Γ is the boundary of G, $R(\lambda)$ is the resolvant of A at λ and

$$\delta = \min_{\lambda \in \Gamma} \frac{1}{||R(\lambda)||} = \min_{\lambda \in \Gamma} \min_{\substack{x \in X \\ ||x||=1}} ||(A - \lambda)x||,$$

one can take

$$\rho = \frac{\delta^2}{\delta + l(\Gamma)/2\pi} \quad (< \delta).$$

Proof. Suppose $B \in \mathcal{B}(X)$ and $||A - B|| < \rho$. Then $\Gamma \subset \rho(B)$ because

$$(B - \lambda) = \big(1 - (A - B)R(\lambda)\big)(A - \lambda).$$

Now,

$$\begin{aligned} ||P_\Gamma(B) - P_\Gamma(A)|| &= \frac{1}{2\pi}\Big|\Big| \int_\Gamma R(\lambda) \sum_{j=1}^{\infty} \big((A - B)R(\lambda)\big)^j d\lambda \Big|\Big| \\ &\leq \frac{l(\Gamma)}{2\pi} \max_{\lambda \in \Gamma} \frac{||A - B|| \, ||R(\lambda)||^2}{1 - ||A - B|| \, ||R(\lambda)||}. \end{aligned}$$

We conclude

$$||P_\Gamma(A) - P_\Gamma(B)|| < 1.$$

Example. (Exercise) Suppose $X = C[0,1]$ and $h(x) = 1$ for $x \in [0,1]$. Let $Af = f(0)h$. What is $\sigma(A)$? Let $G = B(1, r)$ for $r < 1$. What is ρ? First $l(\Gamma) = 2\pi r$.

$$\max_{x \in I} |f(0) - \lambda f(x)| \geq \max \{|f(0)(1 - \lambda)|, ||f(0)| - |\lambda||\}$$

One more homework problem due Nov. 8.

3. Suppose S and T are to disjoint closed-open subsets of $\sigma(A)$. Show,

$$P_S P_T = P_T P_S = 0.$$

The Fredholm Determinant

Suppose E is a Banach space. If $u \in E'\hat{\otimes}E =: F(E)$, let $\tilde{u}$ denote its image in $\mathcal{B}(E)$. We call $\tilde{u}$ nuclear. It is compact. We know $F(E)$ is a Banach algebra, the map $F(E) \to \mathcal{B}(E)$ is continuous and moreover we have a map

$$\mathrm{Tr} : F(E) \to \mathbf{C}.$$

Definition. *Let $u \in F(E)$. The Fredholm determinant,* $\det(1 - zu)$ *of u is*

$$\exp(Tr \log(1 - zu)) = \exp\big(- \sum_{n=1}^{\infty} \; Tr\,(u^n)\frac{z^n}{n}\big).$$

We will prove:

Theorem. *If $u \in F(E)$,* $\det(1 - zu)$ *converges on the entire complex plane. All non-zero complex numbers are are normal points for $\tilde{u}$ and if $\lambda \in \mathbf{C}^*$, $1/\lambda$ is a zero of the Fredholm determinant of u of order n if and only if the radical multiplicity of $\tilde{u}$ at λ is n.*

2. Some linear Algebra

(I generalized vector spaces while giving this lecture to A-modules.) For a vector space E, $\hat{E}$ will denote its dual and $\langle\ ,\ \rangle$ the natural pairing between E and $\hat{E}$. An n-linear map from V to W is a map $f(x_1, \ldots, x_n)$ from V^n to W, which is linear in each variable. It is said to be symmetric if

$$f(x_1, \ldots, x_n) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}),$$

for all $\sigma \in S_n$ the symmetric group on $\{1, \ldots, n\}$. There is a obvious pairing between $\bigotimes^n E$ and $\bigotimes^n \hat{E}$,

$$\langle x_1 \otimes \cdots \otimes x_n, \hat{x}_1 \otimes \cdots \otimes \hat{x}_n \rangle = \langle x_1, \hat{x}_1 \rangle \cdots \langle x_n, \hat{x}_n \rangle,$$

and between $\bigwedge^n E$ and $\bigwedge^n \hat{E}$,

$$\langle x_1 \wedge \ldots \wedge x_n, \hat{x}_1 \wedge \ldots \wedge \hat{x}_n \rangle = \det(\langle x_i, \hat{x}_j \rangle).$$

There exist a canonical homomorphism ϕ_n from $\bigotimes^n E$ into $\bigwedge^n E$,

$$\phi_n(x_1 \otimes \cdots \otimes x_n) = x_1 \wedge \cdots \wedge x_n$$

and one, a_n, the other way,

$$a_n(x_1 \wedge \cdots \wedge x_n) = \frac{1}{n!} \sum_{\sigma \in S_n} \epsilon_\sigma x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}.$$

If F is another vector space and $u_1, \ldots, u_n \in L(E, F)$, we define $u_1 \otimes \cdots \otimes u_n$ in $L(\bigotimes^n E, \bigotimes^n F)$ by

$$x_1 \otimes \cdots \otimes x_n \mapsto u_1(x_1) \otimes \cdots \otimes u_n(x_n),$$

and $u_1 \wedge \cdots \wedge u_n$ in $L(\bigwedge^n E, \bigwedge^n F)$ by

$$u_1 \wedge \cdots \wedge u_n = \frac{1}{n!} \phi_n \circ (u_1 \otimes \cdots \otimes u_n) \circ a_n.$$

(This needs to be modified when one talks about modules (See the notes "The determinant," after lecture 35).) In particular, $u_1 \wedge \cdots \wedge u_n$ takes

$$x_1 \wedge \cdots \wedge x_n \quad \text{to} \quad \frac{1}{n!} \sum_{\sigma \in S_n} \epsilon_\sigma u_1(x_{\sigma(1)}) \wedge \cdots \wedge u_n(x_{\sigma(n)}),$$

and if $u_1 = u_2 = \cdots = u_n = u$, this map takes $x_1 \wedge \cdots \wedge x_n$ to $u(x_1) \wedge \cdots \wedge u(x_n)$.
In general, $u_1 \wedge \cdots \wedge u_n$ is n-linear and symmetric.
If $u_i = \hat{e}_i \otimes f_i$, $\hat{e}_i \in \hat{E}$ and $f_i \in F$, then

$$\bigotimes_i u_i = (\hat{e}_1 \otimes \cdots \otimes \hat{e}_n) \otimes (f_1 \otimes \cdots \otimes f_n)$$

and

$$\bigwedge_i u_i = \frac{1}{n!} (\hat{e}_1 \wedge \cdots \wedge \hat{e}_n) \otimes (f_1 \wedge \cdots \wedge f_n)$$

Lecture 32

Fix a commutative unital ring A. Recall, if E and F are A-modules $L(E,F) := L_A(E,F)$ denotes the A-homomorphisms from E to F and $\hat{E} = L(E,A)$. Let $L_f(E,F) = \hat{E} \otimes F$. Also set $L(E) = L(E,E)$ and $L_f(E) = L_f(E,E)$.

Fundamental Forms

We have a natural map $L_f(E) \to L(E)$ (an injection when E is free) and we have a trace map

$$\operatorname{Tr} L_f(E) \to A$$
$$\sum \hat{e}_i \otimes f_i \mapsto \sum \hat{e}_i(f_i)$$

Last time we made a map from $L(E,F)^n$ to $L(\bigwedge^n E, \bigwedge^n F)$, $(u_1, \ldots, u_n) \mapsto u_1 \wedge \cdots \wedge u_n$ such that if $u_i = \hat{e}_i \otimes f_i$,

$$\bigwedge_i u_i = \frac{1}{n!}(\hat{e}_1 \wedge \cdots \wedge \hat{e}_n) \otimes (f_1 \wedge \cdots \wedge f_n),$$

where

$$(\hat{e}_1 \wedge \cdots \wedge \hat{e}_n)(e_1 \wedge \cdots \wedge e_n) = \det(\langle e_i, \hat{e}_j \rangle).$$

This means $(u_1, \ldots, u_n) \mapsto u_1 \wedge \cdots \wedge u_n$,

$$L_f(E)^n \to L_f(\bigwedge^n E),$$

Definition. *If E is an A-module and n is a positive integer, let the fundamental form α_n on $L_f(E)^n$ be the symmetric linear form defined by*

$$\alpha_n(u_1, \ldots, u_n) = \operatorname{Tr} u_1 \wedge \cdots \wedge u_n.$$

In particular, if $u_i = \hat{e}_i \otimes f_j$,

$$\alpha_n(u_1, \ldots, u_n) = \frac{1}{n!} \det(\langle \hat{e}_i, f_j \rangle).$$

We set $\alpha_n(u) = \alpha_n(u, \ldots, u)$ and $\alpha_0(u) = 1$. We signal, $L(E)$ acts on $L_f(E)$ from both sides and if $A \in L(E)$,

$$\alpha_n(u_1 A, \ldots u_n A) = \alpha_n(A u_1, \ldots, A u_n).$$

The theorem we want is

Theorem. *Suppose E is free of rank n and $u \in L(E)$. Then*

$$\det(1+u) = \sum_{n=0}^{\infty} \alpha_n(u).$$

This latter sum is finite because:

Proposition. *Suppose $u \in L(E,F)$ and $u(E)$ is contained a submodule generate by m elements. Then $\bigwedge^k u = 0$ for all $k > n$.*

Proof. Recall, $\left(\bigwedge^k u\right)(x_1 \wedge \cdots \wedge x_k) = ux_1 \wedge \cdots \wedge ux_k$.
Our form α_n is n-linear form on $L_f(E)$ so we can think of it as a linear form on $\bigotimes^n(\hat{E} \otimes E)$ and so a $2n$-linear form on $E^n \times \hat{E}^n$, where it is $\frac{1}{n!}\det(\langle e_i, \hat{e}_j\rangle)$. This is clearly alternating in the variables $e_1, \ldots, e_n$ and in the variables $\hat{e}_1, \ldots, \hat{e}_n$. Say such a form is bi-alternating.

Lemma. *If f is a n-times linear bi-alternating form on $L_f(E)^n$, there exists a unique linear form g on $\bigwedge^n \hat{E} \otimes \bigwedge^n E$ such that*

$$f(u_1, \ldots, u_n) = g(u_1 \wedge \cdots \wedge u_n).$$

Proposition. *If E is free of finite rank, any n-linear bi-alternating form on $L_f(E)^n$ such that*

$$f(u_1 A, \ldots u_n A) = f(Au_1, \ldots, Au_n) \qquad (\text{ for } A, u_i \in L_f(E))$$

is a multiple of α_n.

Proof. Such an f corresponds to a g as above such that

$$g(BU) = g(UB) \qquad [U = \wedge_i u_i, B = \wedge^n A, A, u_i \in L_f(E)].$$

Homework problems due Nov. 15

Compute ρ for the radical multiplicity example of Monday.

1. Research Problem

Suppose E is Banach space and A is a unital comutative Banach algebra contained in $\mathcal{B}(E)$. Let $E'_A = Hom_A^{Cont}(E, A)$. If $h \in \sigma(A)$ then we set $h(E) = E\hat{\otimes}_h\mathbf{C}$ so we have a homomorphism $E \to h(E)$, $e \mapsto 1 \otimes e$ which we will call h. We also have a homomorphism $E'_A \to h(E)'$ $E'_A \to h(E'_A) \to h(E)'$ which we call h'. For a series, $f(T) = \sum_{n=0}^{\infty} a_n T^n \in A[[T]]$, let $h(f(T)) = \sum_{n=0}^{\infty} h(a_n)T^n$. Then $h' \otimes h$ is a homomorphism from $E'_A \otimes_A E$ to $h(E)' \otimes h(E)$. I'll call this h too.

Problem. *Define a semi-norm on $E'_A \otimes_A E$ so that $h' \otimes h$ is continuous $\forall h \in \sigma(A)$, if $E'_A\hat{\otimes}E$ be the corresponding completion, we will have a trace map*

$$Tr\colon E'_A\hat{\otimes}E \to A.$$

and if for $u \in E'_A\hat{\otimes}E$

$$\det_A(1 - Tu) = \exp\Big(-\sum_{n=1}^{\infty} Tr\,(u^n)\frac{z^n}{n}\Big),$$

then

$$h\Big(\det_A(1{-}zu)\Big) = \det(1 - zh(u)).$$

2. Some Ideas

It's "clear" what semi-norm to put on $E'_A \otimes_A E$. Namely,

$$||u|| = \sup_{h\in\sigma(A)} ||h(u)||.$$

For various reasons I don't think this will work in general. But there is one case I expect it to work, i.e., when E has a "basis" over A. That is, suppose there exist $e_i \in E, i \geq 0$ such that for each $d \in E$ there exist unique $a_i \in A, i \geq 0$ so that

$$\sum_{i=0}^{n} a_i e_i \to d.$$

1. More on the Fundamental Forms

Suppose E is an A-module and n is a positive integer. Recall, we defined the fundamental form α_n on $L_f(E)^n$ by

$$\alpha_n(u_1, \dots, u_n) = \operatorname{Tr} u_1 \wedge \cdots \wedge u_n.$$

Our form α_n is an n-linear form on $L_f(E)$ so a linear form on $\bigotimes_A^n(\hat{E} \otimes_A E)$ and so a $2n$-linear form on $E^n \times \hat{E}^n$, where it is $\frac{1}{n!}\det(\langle e_i, \hat{e}_j\rangle)$. This is alternating in the variables $e_1, \dots, e_n$ and in the variables $\hat{e}_1, \dots, \hat{e}_n$. Say such a form is **bi-alternating**.

Lemma. *If f is a n-times linear bi-alternating form on $L_f(E)^n$, there exists a unique linear form g on $\bigwedge^n \hat{E} \otimes_A \bigwedge^n E$ such that*

$$f(u_1, \dots, u_n) = g(u_1 \wedge \cdots \wedge u_n).$$

Proof. Functorial properties of $\otimes$ and $\wedge$ imply the existence of a unique linear form g on $\bigwedge^n \hat{E} \otimes \bigwedge^n E$ such that if $\hat{e}_i \in \hat{E}$ and $f_i \in F$,

$$f(\hat{e}_1 \otimes f_1, \dots, \hat{e}_n \otimes f_n) = n! \cdot g\big((\hat{e}_1 \wedge \cdots \wedge \hat{e}_n) \otimes (f_1 \wedge \cdots \wedge f_n)\big).$$

Also recall, if $u_i = \hat{e}_i \otimes f_i$, then

$$\bigwedge_i u_i = \frac{1}{n!}(\hat{e}_1 \wedge \cdots \wedge \hat{e}_n) \otimes (f_1 \wedge \cdots \wedge f_n).$$

Lemma. *If E is free of finite rank, any n-linear bi-alternating form on $L(E)^n$ is a multiple of α_n if*

$$f(u_1A, \dots u_nA) = f(Au_1, \dots, Au_n) \qquad \text{(for } A, u_i \in L(E)\text{)}.$$

Proof. Such an f corresponds to a g as above, such that

$$g(BU) = g(UB) \qquad [U = \wedge_i u_i, B = \wedge^n C, C, u_i \in L_f(E)] \qquad (*).$$

The point is that,

$$Cu_1 \wedge \ldots \wedge Cu_n = (\bigwedge^n C)(u_1 \wedge \ldots \wedge u_n).$$

Check when $n = 2$: Suppose $x_1, x_2 \in E$ and $u_i = \hat{a}_i \otimes b_i$. Then, $2(C \wedge C)((u_1 \wedge u_2)(x_1 \wedge x_2))$

$$\begin{aligned}
&= (C \wedge C)\big(\hat{a}_1(x_1)b_1 \wedge \hat{a}_2(x_2)b_2 - \hat{a}_1(x_2)b_1 \wedge \hat{a}_2(x_1)b_2\big) \\
&= \hat{a}_1(x_1)C(b_1) \wedge \hat{a}_2(x_2)C(b_2) - \hat{a}_1(x_2)C(b_1) \wedge \hat{a}_2(x_1)C(b_2) \\
&= 2(Cu_1 \wedge Cu_2)(x_1 \wedge u_2)
\end{aligned}$$

Now it follows that $(*)$ holds true for all $U \in \bigwedge^n \hat{E} \otimes_A \bigwedge^n E$ and for all

$$B = C_1 \wedge \cdots \wedge C_n \qquad (C_i \in L(E)).$$

Hence for all $B \in \bigwedge^n \hat{E} \otimes_A \bigwedge^n E$. Let $F = \bigwedge^n E$. Then from $(*)$, we know g is proportional to the trace form which is what we wanted. ∎

Proposition. *This conclusion of the lemma will hold if we can write $E = M \oplus N$ with the properties that $u_i(M) = 0$, $u_i(E) \subseteq N$ and N is free of finite rank.*

Proof. Just restrict to N.

This is always the case when $A = \mathbf{C}$.

1. More on the Fundamental Forms

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Lemma. *If f is a n-times linear bi-alternating form on $L_f(E)^n$, there exists a unique linear form g on $\bigwedge^n \hat{E} \otimes_A \bigwedge^n E$ such that*

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Lemma. *If E is free of finite rank, any n-linear bi-alternating form on $L(E)^n$ is a multiple of α_n if*

$$f(u_1 A, \ldots u_n A) = f(Au_1, \ldots, Au_n) \qquad \text{(for } A, u_i \in L(E)\text{)}.$$

2. Taylor series for $\det(1+u)$.

Theorem. *Suppose E is free of rank m and $u \in L(E)$. Then*

$$\det(\mathbf{1} + u) = \sum_{n=0}^{\infty} \alpha_n(u). \qquad (**)$$

Proof. First

$$\alpha_m(u) = \det(u).$$

Also,

$$\alpha_n(\mathbf{1}) = \binom{m}{n}.$$

Lemma. *Suppose $p \leq m$. Then.*

$$\alpha_n(u_1, \ldots, u_p, \underbrace{\mathbf{1}, \ldots, \mathbf{1}}_{n-p}) = \left(\binom{m}{n} \Big/ \binom{m}{p}\right) \alpha_p(u_1, \ldots, u_p)$$

Proof of Theorem.

Now if E is not free of finite rank but $u \in L_f(E)$, the right hand side of (∗∗) still makes sense and this we'll take as a definition of the left hand side in this case. (Warning: we may not be able to identify $L_f(E)$ with a sub-module of $L(E)$ in general.)

Call a direct summand of E which is free of finite rank a DSFR. If N is a **DSFR** and $u(E) \subseteq N$, then

$$\det(\mathbf{1} + u) = \det((\mathbf{1} + u)|_N). \qquad (exercise)$$

(This is false in general (see note below, "The determinant").) So in particular, if u, v and uv all have images inside the same DSFR, (this is false in general) then

$$\det(1 + u) \det(1 + v) = \det\big((1 + u)(1 + v)\big)$$

Homework problems due Fri. Nov. 22

A. Do the exercise above.

B. Generalize Théorème 1, Chapitre II of La Théorie de Fredholm to our context of "Banach A-modules."

ermail note:

Dear Class,

Mala pointed out to me that problem A is false as stated. It is true when $L_f(E)$ injects into $L(E)$ and in general the two sides differ by an invertible element of A. Prove this instead. To prove the second of these two assertions, assume

$$det(1 + u)det(1 + v) = det((1 + u)(1 + v)),$$

for u and v in $L_f(E)$, which we will prove.

Also, to convince you that we are not just barking at the wind, a good example of a Banach A-module is: the module $C(X, E)$ over the ring $A = C(X)$, where X is a compact Hausdorff space, E is a Banach space and $C(X, E)$ is the space of continuous functions from X into E. See you tomorrow,

Robert

The Determinant

Suppose A is a commutative unital $\mathbf{Q}$-algebra. Let $\mathcal{T}$ be the category of triples, $(M, N, (\ ,\ ))$ where M and N are A modules and

$$(\ ,\ ) \colon M \times N \to A$$

is a A-bilinear pairing, with the obvious morphisms. Supppose $T = (M, N, (\ ,\ )) \in \mathcal{T}$. Then we can make an A-algebra R_T out of $M \otimes N$ by setting,

$$(m \otimes n) \cdot (l \otimes p) = (l, n) m \otimes p.$$

We also have an A-module homomorphism $\mathrm{Tr} \colon R_T \to A$ determined by

$$\mathrm{Tr}\,(a \otimes b) = (a, b).$$

We can make a monoid G_T out of R_T by setting

$$f * g = f + g + fg.$$

We will label the element in G_T corresponding to $a \in R_T$, $\mathbf{1} + a$. We wish to make a homomorphism from G_T into the monoid A under multiplication,

$$\det_T \colon G_T \to A.$$

We have a symmetric A-module n-linear map $R_T^n \to \bigwedge^n M \otimes \bigwedge^n N$ written $(u_1, \ldots, u_n) \mapsto u_1 \wedge \cdots \wedge u_n$ described on decomposable elements as follows: $(\hat{e}_1 \otimes d_1, \ldots, \hat{e}_n \otimes d_n)$ goes to

$$\frac{1}{n!} \hat{e}_1 \wedge \cdots \wedge \hat{e}_n \otimes d_1 \wedge \cdots \wedge d_n.$$

We also have a pairing between $\bigwedge^n M$ and $\bigwedge^n N$ defined by

$$(\hat{e}_1 \wedge \cdots \wedge \hat{e}_n, e_1 \wedge \cdots \wedge e_n)_n = \det((\hat{e}_i, e_j))$$

and so we have an element $\bigwedge^n T := \left(\bigwedge^n M, \bigwedge^n N, (\ ,\ )_n\right)$ of $\mathcal{T}$ and an A-module homomorphism

$$\mathrm{Tr}\colon \bigwedge^n M \otimes \bigwedge^n N \to A.$$

Now, we define, for $u_i \in R_T$,

$$\alpha_{T,n}(u_1, \ldots, u_n) = \ \mathrm{Tr}\,(u_1 \wedge \cdots \wedge u_n)$$

and if $u \in R_T$ we put $\alpha_{T,n}(u) = \alpha_{T,n}(u \wedge \cdots \wedge u)$, $n > 0$, $\alpha_{T,0}(u) = 1$ and we set

$$\det_T(\mathbf{1} + u) = \sum_{n=0}^{\infty} \alpha_{T,n}(u).$$

(This is a finite sum.) Claim: $\det_T\colon G_T \to A$ is a homomorphism of monoids.
By what we did in class, we know $\det_T$ is a homomorphism (honestly, the determinant) when M is free of finite rank over A, $N = \hat{M}$ and $(\ ,\ )$ is the natural pairing. (In this case, R_T is the ring of $n \times n$ matrices over A.) We will prove the general case, by using this and functoriality.
Clearly, the associations $T \mapsto R_T$ and $T \mapsto G_T$ are functorial, $T \mapsto \alpha_{T,n}$ is functorial as well and so is $T \mapsto \det_T$. Suppose $u = \sum_{i=1}^n a_i \otimes b_i$ and $v = \sum_{i=1}^m c_i \otimes d_i$ are elements of R_T. Let F be the free A-module on the symbols $x_1, \ldots, x_n$ and $y_1, \ldots, y_m$. Let $f\colon F \to M$ be the A-module homomorphism determined by $f(x_i) = a_i$ and $f(y_i) = c_i$ and let S be the element of $\mathcal{T}$, $(F, N, (\ ,\ )')$ where

$$(x, n)' = (f(x), n).$$

Clearly there is a morphism ρ from S to T which takes

$$u' = \sum_{i=1}^{n} x_i \otimes b_i \quad \text{and} \quad v' = \sum_{i=1}^{m} y_i \otimes d_i$$

to u and v respectively and so

$$\det_S(\mathbf{1} + u') = \det_T(\mathbf{1} + u), \quad \det_S(\mathbf{1} + v') = \det_T(\mathbf{1} + v) \ \text{ and}$$
$$\det_S\big((\mathbf{1} + u')(\mathbf{1} + v')\big) = \det_T((\mathbf{1} + u)(\mathbf{1} + v)\big).$$

Now there is also an obvious morphism from S to $T' = (F, \hat{F}, \langle\ ,\ \rangle)$ where $\langle\ ,\ \rangle$ is the natural pairing. Let u'' denote the image of u' and v'' the image of v' in $R_{T'}$. Then,

$$\det_S(\mathbf{1}+u') = \det_{T'}(\mathbf{1}+u''), \quad \det_S(\mathbf{1}+v') = \det_{T'}(\mathbf{1}+v'') \text{ and}$$
$$\det_S\big((\mathbf{1}+u')(\mathbf{1}+v')\big) = \det_{T'}((\mathbf{1}+u'')(\mathbf{1}+v'')\big).$$

The claim now follows from the fact that $\det_{T'}: G_{T'} \to A$ is a homomorphism.

Examples:

(i) Suppose $A = \mathbf{C}[x]/x^2$, $M = xA$ and $N = \hat{M}$ with the natural pairing. Then $N \cong M$ and generated by the homomorphism $\rho: M \to A$, $m \mapsto m$. The module $M \otimes N \neq 0$ (in fact, is isomorphic to M and generated by $u := x \otimes \rho$) but the natural map from $M \otimes N$ to $End_A(M)$ is 0. We note that $\operatorname{Tr} u = x$ and $\det(\mathbf{1}+u) = 1+x$.
(ii) Now let $A = \mathbf{C}[a,b,c,d]/(a^2+bc, d^2+bc, (a+d)b, (a+d)c)$. (Think of $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and suppose $B^2 = 0$.) Let M be the module

$$\frac{Ae_1 + Ae_2}{(ae_1 + ce_2, be_1 + de_2)}.$$

(Think of the column vectors $\begin{pmatrix} a \\ c \end{pmatrix}$ and $\begin{pmatrix} b \\ d \end{pmatrix}$. (Do these generate the module of $v \in A^2$ such that $Bv = 0$?)) There are elements f_1 and $f_2 \in \hat{M}$ determined by

$$f_1(e_1) = a \quad \text{and} \quad f_1(e_2) = b$$
$$f_2(e_1) = c \quad \text{and} \quad f_2(e_2) = d$$

Let $u = f_1 \otimes e_1 + f_2 \otimes e_2 \in \hat{M} \otimes M$. Then the image of u in $End_A(M)$ is 0, $\operatorname{Tr} u = a+d$ and $\det(\mathbf{1}+u) = 1 + (a+d) + (ad-bc)$. One can show $(a+d)^3 = 0$ and $(ad-bc)^2 = 0$ but, $(a+d)^2 \neq 0$.
If $u \in M \otimes \hat{M}$ and $u^2 = 0$, I think, $\operatorname{Tr} u$ is nilpotent.

The Exponential and the Determinant

Denote the $\mathcal{T}$ in the section "The determinant" by $\mathcal{T}_A$ (we will be changing rings). We will say an element $T = (M, N, (\ ,\ )) \in \mathcal{T}_A$ is free of finite rank if M is free of finite rank, $N = \hat{M}$ and $(\ ,\ )$ is the natural pairing.

Suppose $T = (M, N, (\ ,\ )) \in \mathcal{T}_A$. First, we can make an element
$T(z) = (M(z), N(z), (\ ,\ )_z)$ of $\mathcal{T}_{A[z]}$ by setting

$$M(z) = M \otimes_A A[z], \quad N(z) = N \otimes_A A[z]$$

and extending $(\ ,\ )$ in the obvious way. If $u \in R_T$, then $zu \in R_{T(z)}$.

Proposition. *If $u \in \mathbf{R}_T$,*

$$\det(1 - zu) = \exp\big(-\sum_{n=1}^{\infty} Tr(u^n)\frac{z^n}{n}\big). \qquad (*)$$

Proof. If B is a square matrix over $\mathbf{C}$

$$\det\big(\exp(B)\big) = \exp(\mathrm{Tr}\, B).$$

which one proves by checking it is true for upper triangular matrices and is well behaved under conjugation. It follows that it is true for square matrices B over $\mathbf{C}[z]$ and then after completing z-adically for square matrices B over $\mathbf{C}[[z]]$. Now suppose, $A = \mathbf{C}$ and $T \in \mathcal{T}_A$ is free of finite rank. Let $u \in R_T$. Set $B = \log(1 - zu)$ (thinking of u as a square matrix) and we deduce $(*)$ in this case. Now, still in this case, we may regard the coefficients on both sides of $(*)$ as polynomials over $\mathbf{Q}$ in the entries in the matrix corresponding to u. From this we deduce that $(*)$ is true for any $T \in \mathcal{T}_A$ free of finite rank over any $\mathbf{Q}$-algebra A. The result, in the general case, follows by functoriality. ∎

1. Determinants

Definition.. *Suppose E is an A-module and and $u \in L_f(E)$. Then we set*

$$\det(\mathbf{1}+u) = \sum_{n=0}^{\infty} \alpha_n(u). \qquad (**)$$

When E is free of finite rank this is a true formula.

Call a direct summand of E which is free of finite rank a DSFR. If N is a **DSFR** and $u(E) \subseteq N$, then

$$\det(\mathbf{1}+u) = \det((\mathbf{1}+u)|_N). \qquad (exercise)$$

So in particular, if u, v and uv all have images inside the same DSFR, then

$$\det(1+u)\det(1+v) = \det\big((1+u)(1+v)\big)$$

2. Banach A-modules.

Suppose E is a Banach space and A is commutative unital Banach algebra which acts continuously on E. Call such a thing a **Banach A-module**. Suppose that E' is the submodule of $\hat{E}$ consisting of continuous A-module homomorphisms $E \to A$. For $u \in E_1 \otimes_A \cdots \otimes_A E_n$, set

$$||u|| = \inf\left(\sum_{i=1}^{m} |y_{i\,1}|\cdots|y_{i\,n}|\right),$$

where the infimum is taken over all representations

$$y_{1\,1} \otimes \cdots \otimes y_{1\,n} + \cdots + y_{m\,1} \otimes \cdots \otimes y_{m\,n} = u.$$

Let $E_1 \hat{\otimes}_A \cdots \hat{\otimes}_A E_n$ denote the corresponding completion.

3. The Fredholm Determinant

Proposition. *On* $(E' \otimes_A E)^n$,

$$||\alpha_n|| \leq \frac{k_n}{n!},$$

where

$$k_n = \sup_{\substack{(x_i)\in E^n, (x'_i)\in E'^n \\ ||x_i||=||x'_j||=1}} ||\det(\langle x_i, x'_j \rangle)||.$$

Proof.

Theorem. *Suppose the Gelfand transform of* A *isometric. Then, the* n*-times linear map* α_n *on* $(E' \otimes E)^n$ *extends by continuity to* $(E' \hat{\otimes} E)^n$ *and the series*

$$\sum_{n=0}^{\infty} \alpha_n(u).$$

is absolutely convergent on $E' \hat{\otimes} E$.

Proof. Use Hadamard and Stirling: $n! \sim \sqrt{2\pi n}\big(\frac{n}{e}\big)^n$.

Corollary. *The series* $\det(1 - Tu)$ *is entire in* T.

1. The Fredholm Resolvant

For $u \in E'\hat{\otimes}E$ set $R(u,T) =$

$$\det(1-uT)/(1-uT) = 1 + \big(u - \alpha_1(u)\big)T + \cdots + b_n(u)T^n + \cdots$$

where $b_n(u) = b_{n-1}(u)u - \alpha_n(u)$ and $b_0(u) = 1$.

Theorem. *If $A = \mathbf{C}$ or (?), the series $R(u,T)$ is entire in T.*

Corollary. *If $A = \mathbf{C}$, the set of inverses of non-zero elements in the spectrum of the image of u in $\mathcal{B}(E)$ is the zero locus of $\det(1-Tu)$.*

2. Analysis of the Fredholm resolvant.

Assume: If $w \in E$ and $h(w) = 0$ for all $h \in \hat{E}$, then $w = 0$.

Proposition/Definition. *Given $u_1, \ldots, u_n \in L_f(E)$ there exists an $R_n(u_1,\ldots,u_n) \in L(E)$ determined by the identity*

$$Tr(vR_n(u_1,\ldots,u_n)) = (n+1)\alpha_{n+1}(u_1,\ldots,u_n,v).$$

Set $R_n(u) = R_n(u,\ldots,u)$.

Proof. We can suppose $u_i = x_i' \otimes x_i$ and $v = x' \otimes x$. Set $x_{n+1} = x$ and $x_{n+1}' = x'$ and let $M = (\langle x_i, x_j\rangle)_{1\le i,j\le n}$. Then

$$\begin{aligned}(n+1)!\alpha_{n+1}(u_1,\ldots,u_n,v) &= \det(\langle x_i, x_j\rangle)\\ &= \det M\,\langle x, x'\rangle + \sum_{1\le i,j\le n} \mu_{ij}\langle x_i, x'\rangle\langle x, x_j'\rangle.\end{aligned}$$

Now set

$$R_n(u_1,\ldots,u_n)\ \text{“=”}\ \frac{1}{(n+1)!}\det\begin{pmatrix} & & & x_1\\ & M & & \vdots\\ & & & x_n\\ x_1' & \ldots & x_n' & 1\end{pmatrix}$$

Proposition. *For* $\lambda \in A$,

$$R(u, -\lambda) = \sum_{n=0}^{\infty} R_n(u)\lambda^n.$$

Lemma. *If* $u_1, \ldots, u_n$ *and* v *are elements of* $L_f(E)$, $(n+1)\alpha_{n+1}(u_1, \ldots, u_n, v) =$

$$-\sum_{i=1}^{n} \alpha_n(u_1, \ldots, u_{i-1}, y_i, u_{i+1}, \ldots, u_n) + \alpha_n(u_1, \ldots, u_n) \ \mathrm{Tr}\, v$$

where y_i *can be either* $u_i v$ *for all* i *or* $v u_i$ *for all* i.

Proof. Wolog $u_i = \hat{x}_i \otimes x_i$ and $v = \hat{x} \otimes x$. Expand by minors. The key point is that

$$u_i v = \langle x, \hat{x}_i \rangle \hat{x} \otimes x_i \quad \text{and} \quad \langle x_i, \hat{x} \rangle \hat{x}_i \otimes x.$$

Corollary.

$$R_n(u) = \alpha_n(u) - uR_{n-1}(u) = \alpha_n(u) - R_{n-1}(u)u.$$

1. Analysis of the Fredholm Resolvant (continued)

Assume: If $w \in E$ and $h(w) = 0$ for all $h \in \hat{E}$, then $w = 0$.

Proposition/Definition. *Given $u_1, \ldots, u_n \in L_f(E)$ there exists an $R_n(u_1, \ldots, u_n) \in L(E)$ determined by the identity*

$$Tr(vR_n(u_1, \ldots, u_n)) = (n+1)\alpha_{n+1}(u_1, \ldots, u_n, v).$$

Set $R_n(u) = R_n(u, \ldots, u)$.

Proposition.

$$R(u, -T) = \det(1 + uT)/(1 + uT) = \sum_{n=0}^{\infty} R_n(u)T^n.$$

Lemma. *If $u_1, \ldots, u_n$ and v are elements of $L_f(E)$, $(n+1)\alpha_{n+1}(u_1, \ldots, u_n, v) =$*

$$-\sum_{i=1}^{n} \alpha_n(u_1, \ldots, u_{i-1}, y_i, u_{i+1}, \ldots, u_n) + \alpha_n(u_1, \ldots, u_n)\ \mathrm{Tr}\, v$$

where y_i can be either $u_i v$ for all i or $v u_i$ for all i.

Proof. Wolog $u_i = \hat{x}_i \otimes x_i$ and $v = \hat{x} \otimes x$. Expand by minors. The key point is that

$$u_i v = \langle x, \hat{x}_i \rangle \hat{x} \otimes x_i \quad \text{and} \quad \langle x_i, \hat{x} \rangle \hat{x}_i \otimes x.$$

Corollary.

$$R_n(u) = \alpha_n(u) - uR_{n-1}(u) = \alpha_n(u) - R_{n-1}(u)u.$$

2. Entirety of Fredholm Resolvant

Suppose now A is a commutative unital Banach-Algebra. To prove the Fredholm resolvant series is entire we need an assumption in addition to that the Gelfand transform is isometric:

Definition. *Call a Banach A-module E with the following property a* **Hahn-Banach module**. *If $w \in E$,*

$$||w|| = \sup\{||h(w)|| : h \in Hom_{A,cont.}(E, A), ||h|| = 1\}.$$

Proposition. *If the Gelfand-Transform of A is isometric and E is a Hahn-Banach A-module, then the n-times linear map R_n on $(E' \otimes_A E)^n$ to $L(E)$ extends by continuity to $(E' \hat{\otimes}_A E)^n$ and the series*

$$\sum_{n=0}^{\infty} R_n(u)$$

is absolutely convergent on $E' \hat{\otimes} E$.

Homework problems due Monday, Dec. 1

A. Suppose E is a Banach space and X is a compact Hausdorff space. Let $A = C(X)$ and $M = C(X, E)$ denote the set of continuous maps of X into E. (i) Show the Gelfand transform of A is isometric. (ii) Show M is a Hahn-Banach A-module with norm, the sup norm. (iii) If U is a continuous map from X into $\mathcal{B}(E)$, then the map

$$N_U(f)(x) = U(x)(f(x))$$

is an operator on M over A. Suppose V is a nuclear operator on E and $VU = (x \mapsto VU(x))$. Show N_{VU} is nuclear over A (i.e. in the image of $M' \hat{\otimes}_A M$). (iv) If $U(x)$ is in nuclear for all x, is N_U nuclear over A? If not, find additional hypotheses to force it to be.

1. Entirety of Fredholm Resolvant

Suppose now A is a commutative unital Banach-Algebra.

Definition. *Call a Banach A-module E with the following property a* **Hahn-Banach module**. *If $w \in E$,*

$$||w|| = \sup\{||h(w)|| : h \in Hom_{A,cont.}(E, A), ||h|| = 1\}.$$

Assume from now on that the Gelfand transform of A is isometric and E is a Hahn-Banach A-module. We know: Given $u_1, \ldots, u_n \in L_f(E)$ there exists an $R_n(u_1, \ldots, u_n) \in L(E)$ determined by the identity

$$Tr(vR_n(u_1, \ldots, u_n)) = (n+1)\alpha_{n+1}(u_1, \ldots, u_n, v).$$

Moreover, if $R_n(u) = R_n(u, \ldots, u)$,

$$R(u, -T) = \det(1 + uT)/(1 + uT) = \sum_{n=0}^{\infty} R_n(u)T^n.$$

Proposition. *If the Gelfand-Transform of A is isometric and E is a Hahn-Banach A-module, then the n-times linear map R_n on $(E' \otimes_A E)^n$ to $L(E)$ extends by continuity to $(E'\hat{\otimes}_A E)^n$ and if $u \in E'\hat{\otimes} E$, the series*

$$\sum_{n=0}^{\infty} R_n(u)$$

is absolutely convergent.

This follows from,

Lemma. *On $(E' \otimes_A E)^n$,*

$$||R_n|| \leq \frac{k_{n+1}}{n!},$$

where

$$k_m = \sup_{\substack{(x_i)\in E^m, (x_i')\in (E')^m \\ ||x_i|| = ||x_j'|| = 1}} ||\det(\langle x_i, x_j' \rangle)||.$$

2. The Fredholm Determinant and the Spectrum

Set $L^*(E) = E'\hat{\otimes}E$. Suppose $u \in L^*(E)$ and $\tilde{u}$ is its image in $\mathcal{B}(E)$. Let $L^e(E)$ denote the ring $L^*(E)$ with the unit $\mathbf{1}$ adjoined.

Lemma. *(i) The image of $L^*(E)$ in $\mathcal{B}(E)$ is an ideal. (ii) If $u \in L^*(E)$ and $\tilde{u} = 0$, then $\mathbf{1} - u$ is invertible in $L^e(E)$. (ii) The element $1 - \tilde{u}$ is invertible in $\mathcal{B}(E)$ if and only if $\mathbf{1} - u$ is invertible in $L^e(E)$.*

Proof. Suppose $1 - \tilde{u}$ is invertible. Write its inverse in the form $1 - v$. Then $v = -\tilde{u} + \tilde{u}v$ so $v = \tilde{w}$ for some $w \in L^*(E)$. It follows that the image of $-u - w + uw$ in $\mathcal{B}(E)$ equals 0 and hence $(\mathbf{1} - u)(\mathbf{1} - w)$ is invertible in $L^e(E)$.

Proposition. *Suppose $a \in A$. Then, $1 - a\tilde{u}$ is invertible in $\mathcal{B}(E)$ if and only if $P_u(a) = \det(\mathbf{1} - au)$ is invertible in A.*

Proof. We know $(\mathbf{1} - au)R(u, a) = P_u(a)$. Now suppose $1 - a\tilde{u}$ is invertible. Then we know $\mathbf{1} - au$ has an inverse $\mathbf{1} - v$ in $L^e(E)$ and

$$\det(\mathbf{1} - au)\det(\mathbf{1} - v) = 1.$$

Homework problems due Due Wed. Dec. 4

B. Suppose M is an A-Banach module. For $\omega \in \sigma(A)$, let $M_\omega = M/\overline{I_\omega M} =: M\hat{\otimes}_\omega \mathbf{C}$, where $I_\omega \subset A$ is the kernel of ω. For $d \in M$, let $\omega(d)$ denote its image in M_ω and for $e \in M_\omega$, put $||e||_\omega = \min\{||d|| : d \in M, \omega(d) = e\}$. Show: (i) M_ω with this norm is a Banach space. (ii) If the Gelfand transform of A is isometric and M is a Hahn-Banach module then, for $w \in M$, $||w|| = \sup\{||\omega(w)||_\omega : \omega \in \sigma(A)\}$. (iii) Show that in the $M = C(X, E)$ example of problem A, $M_\omega \cong E$ for all ω.

Riesz Theory

Now we follow Serre's *Endomorphismes complètements continues des espaces de Banach p-adiques, Publ. Math. I.H.E.S.*, **12** (1962) 69-85.

Let $D = \frac{d}{dT}$. We say a is a zero of $H(T)$ in A of order h if $D^s H(a) = 0$ for $s < h$ and $D^h H(a) \in A$ is invertible.

Let $u \in L^*(E)$ and set $P(T) = \det(1 - Tu)$ and $R(T) = R(u, T)$.

Proposition. *Suppose $a \in A$ is a zero of $P(T)$ of order h. Then we have a unique decomposition*

$$E = N(a) \oplus F(a)$$

into closed submodules such that $1-au$ is invertible on $F(a)$ and $(1-au)^h N(a) = 0$.

Proof. From $(1 - uT)R(T) = P(T)$, we get

$$(1 - Tu)D^n R(T) - nuD^{n-1} R(T) = D^n P(T).$$

Suppose for simplicity, $h = 2$. And let $c = D^2 P(a)$. Then,

$$(1 - au)R(a) = (1 - au)DR(a) - uR(a) = 0$$
$$(1 - au)D^2 R(a) - 2uDR(a) = c.$$

Set

$$e = c^{-1}(1 - au)D^2 R(a), \quad \text{and} \quad f = -c^{-1}uDR(a).$$

Then $e + f = 1$ and $fe^2 = 0$. Now define projectors $p = e^2$ and $q = 2ef + f^2$.

Corollary. *When $A = \mathbf{C}$, $\dim N(a) = h$.*

We will need,

Proposition (Proposition 2 of Grothendieck)**.** *Suppose* $A = \mathbf{C}$ *and* E *can be expressed as a direct sum* $M_1 \oplus M_2$ *of closed subspaces and* p_1 *and* p_2 *are the corresponding projections. Then* $L^*(E_i)$ *is naturally isomorphic to* $p_i L^*(E) p_i$ *and if* $uM_1 \subseteq M_1$*,*

$$\det(1+u) = \det(\mathbf{1} + u_{M_1}) \det(\mathbf{1} + u_{M_2}) \qquad (u_{M_i} = p_i u p_i)$$

Proof of Corollary.

Suppose M_1 is a finite dimensional subspace of $N(a)$ left invariant by u, say of dimension m. Then, using the Hahn-Banach theorem, there exists a closed complement M_2. Let u_1 and u_2 be as in the proposition . Since M_1 is finite dimensional and $(1-au)^h M_1 = 0$, $L^*(M_1) \cong End(M_1)$ and $\det(1-u_1T) = (1-aT)^m$. Thus $(1-aT)^m | P(T)$ and so $m \le h$ and so $n := \dim(N(a)) \le h$.Also, using the proposition with respect to the decomposition:

$$E = N(a) \oplus F(a)$$

$$P(T) = (1-aT)^n \det(1 - au_{F(a)}).$$

But since $1 - a\tilde{u}$ is invertible on $F(a)$, $n = h$.

I think one can prove more gernerally:

Theorem. *Suppose* $P_u(T) = Q(T)S(T)$ *where* $Q(T)$ *is a polynomial of degree* d *whose leading coefficient is a unit,* $Q(0) = S(0) = 1$ *and* $(Q(T), S(T)) = 1$*. Then there is a direct sum decomposition*

$$E = N \oplus F$$

into closed submodules such that $Q^*(u)N = 0$*,* $Q^*(u)$ *is invertible on* F *and* N *is projective of rank* d*, where* $Q^*(T) = T^d Q(T^{-1})$*.*

1. Hilbert-Schmidt and Nuclear Operators

(See Dunford and Schwartz, Vol. 2 XI.6 & XI.9 and Gohberg and Krejn Chapt. III §8-10.) Suppose (X, μ) is a pair consisting of a locally compact Hausdorff space X and a positive Borel measure μ. Let $H = L^2(X, \mu)$. Then if $k(x, y) \in L^2(X \times X, \mu \times \mu)$ we have defined an operator K on H by

$$Kf(x) = \int_X k(x, y) f(y) d\mu(y). \qquad (*)$$

These are called Hilbert-Schmidt operators.

Theorem. *The set of Hilbert-Schmidt operators is the $*$-ideal of operators K in $\mathcal{B}(H)$ characterised by the property that for an orthonormal "basis" B of H the sum*

$$||K||_2 := \big(\sum_{h \in B} ||Kh||^2\big)^{1/2}$$

is finite. In fact, all Hilbert-Schmidt operators are compact and the above sum is independant of the choice of B.

We also have defined nuclear operators to be those operators in the image of $H' \hat{\otimes} H$.

Lemma. *The map $H' \hat{\otimes} H \to \mathcal{B}(H)$ is an injection.*

Theorem. *The product of any two Hilbert-Schmidt operators is nuclear and if K is a Hilbert-Schmidt operator,*

$$||K||_2 = \sqrt{\mathit{Tr}(K^* K)}.$$

Proposition. *If $X = [a, b]$, $\mu = dt$ and $k(x, y)$ is continuous then the operator K defined in $(*)$ is nuclear if it is self-adjoint and positive and if it is nuclear*

$$\mathrm{Tr}\, K = \int_a^b k(x, x) d\mu(x).$$

2. Classes of Compact Operators

If T is a compact operator on a Hilbert space H, so is T^*T in fact it has a positive square root A. Let

$$\mu_1(T) \geq \mu_2(T) \geq \cdots$$

be the points in the spectrum of A listed with multiplicty. We set

$$||T||_p = \big(\sum_{i=1}^{\infty} \mu_i(T)^p\big)^{1/p},$$

for $p < \infty$, $||T||_\infty = ||T||$ and define S_p to be the set of compact operators T such that $||T||_p < \infty$.

Theorem. *The set S_1 is the set of nuclear opeartors and S_2 is the set of Hilbert-Schmidt operators. Moreover, if $1/p + 1/q = 1/r$, $S_p S_q \subseteq S_r$. In particular, The sets S_p are ideals,*

3. Nuclear Operators

One important result we need is:

Proposition. *If Z is a compact operator there exists orthonormal sets $\{e_i\}$ and $\{f_i\}$ such that*

$$\sum_i \mu_i(Z)(\ , e_i) f_i = Z.$$

Idea of Proof.

Using the polar decomposition, we can write $Z = UP$, where U is a partial isometry and $P = (Z^*Z)^{1/2}$. Now apply the spectral theorem to P.

This is called the Schmidt series for Z.

Definition. *An operator A on H is said to have finite matricial trace if and only if the sum,*

$$\sum (A\chi_i, \chi_i)$$

is finite for any orthonormal basis $\{\chi_i\}$ *of* H.

Theorem. *An operator* $A \in \mathcal{B}(H)$ *has a finite matricial trace if and only if it is in* S_1. *Moreover, the sum*

$$\sum (A\chi_i, \chi_i)$$

is independant of the choice of the basis.

Lemma. *Suppose* B *is a self-adjoint positive operator. The sums,* $M_\chi = \sum_i (B\chi_i, \chi_i)$, *where* $\{\chi_i\}$ *is an orthonormal basis, are independant of the choice of the basis and* B *has finite matricial trace if and only if* $B \in S_1$.

Proof. Let $C = B^{1/2}$. If $\{\phi_j\}$ is another orthonormal basis

$$M_\chi = \sum_{k=1}^{\infty} ||C\chi_k||^2 = \sum_k \sum_j |(C\chi_k, \phi_j)|^2 =$$

Proof of Theorem.

We use the Schmidt series for A. In fact, we claim all these sums equal

$$\sum_j \mu_j(A)(f_i, e_i)$$

Call the sum in the theorem $\operatorname{Tr} A$.

Corollary. *The map* $H' \hat{\otimes} H \to \mathcal{B}(H)$ *is an injection and its image is* S_1.

Proof. We have a continuous map from $H' \times H$ into S_1:

$$u := (\langle\ , h\rangle, g) \mapsto L_u \colon v \mapsto \langle v, h\rangle g.$$

The trace norm of L_u is $||h|| \cdot ||g||$. This gives us a map from $H' \hat{\otimes} H$ to S_1 of norm at most 1. To go the other way, we use the Schmidt series.

Gohberg-Krejn Win

Gohberg-Krejn prove (Dunford-Schwartz do also) that all the $|| \ ||_p$ are norms.

Lemma. $|| \ ||_p \geq || \ ||$.

Lemma. *Suppose $A \in \mathcal{B}_0(H)$ and $\sum_j s_j(A)(\ , e_j) f_j$ is its Schmidt series. Let $B \in \mathcal{B}(H)$ and $\{\chi_i\}$ be any orthonormal basis. Then*

$$\sum_i (AB\chi_i, \chi_i) = \sum_i (BA\chi_i, \chi_i) = \sum_i s_i(A)(Bf_i, e_i).$$

Proof.

$$AB = \sum_j s_j(A)(\ , B^* e_j) f_j$$

$$BA = \sum_j s_j(A)(\ , e_j) B f_j$$

and

$$\sum_i (\sum_j s_j(\chi_i, b_j) a_j, \chi_i) = \sum_j s_j \sum_i (a_i, \chi_i)(\chi_i, b_i).$$

Call, $\sum_i (C\chi_i, \chi_i)$, $\operatorname{Tr} C$ (which could be ∞).

Corollary. *If A and B are as in the proposition, $Tr\, AB = Tr\, BA$ and $Tr\, A^* = \overline{Tr\, A}$.* ∎

Proposition.

$$||A||_1 = \sup_{||B|| \leq 1} | \ Tr\, AB|. \qquad (*)$$

Proof. Let $U|A|$ be the polar decomposition of A. If $B = U^*$, the right hand side of $(*)$ equals $\operatorname{Tr} |A| = ||A||_1$.

Corollary. *(i) $|| \ ||_1$ is a norm. (ii) $||A^*||_1 = ||A||_1$. (iii) S_1 is a $*$-ideal complete w.r.t. $|| \ ||_1$. (iv) $H' \hat{\otimes} H \to \mathcal{B}(H)$ is an injection onto S_1.*

Hilbert-Schmidt operators

Lemma. *An element $A \in \mathcal{B}_0(H)$ is an element of S_2 if and only if $A^*A \in S_1$ and so S_2 is a $*$-ideal.*

Corollary. *If $A, C \in S_2$, $AC \in S_1$.*

Proof. Let $B = C^*$. Then

$$(A+B)^*(A+B) - A^*A - B^*B = A^*B + B^*A$$
$$(A+iB)^*(A+iB) - A^*A - B^*B = iA^*B - iB^*A$$

Theorem. *Suppose $H = L^2(X, \mu)$ and $k(x, y) \in L^2(X \times X, \mu \times \mu) =: H^2$. Then the operator on H*

$$Kf(x) = \int_X k(x, y) f(y) d\mu(y)$$

is in $S_2(H)$ and $||K||_2 = (k, k)_2$.

Proof. Let $\{\phi_i\}$ be an on basis for H. Let $f_{ij} = \phi_i(x)\overline{\phi_j(y)}$. Then $\{f_{ij}\}$ is an on basis for H^2. Then

$$(k, k)_2 = \sum_i |(k, f_{ij})_2|^2,$$

and

$$\begin{aligned}(k, f_{ij})_2 &= \int_{X \times X} \overline{\phi_k(x)} k(x, y) \phi_j(y) dx dy \\ &= (K\phi_j, \phi_k).\end{aligned}$$

Proposition. *Suppose $(X, \mu) = ([a, b], dt)$, $k(x, y)$ is continuous and K is self-adjoint and positive. Then K is nuclear and*

$$\operatorname{Tr} K = \int_a^b k(x, x) dt.$$

Idea of proof. Let $\phi_i(x)$ be an orthonormal set corresponding to $s_i(K)$. Then

$$k(x, y) = \sum_i s_i(K) \phi_i(x) \overline{\phi_i(y)}.$$

Odds and Ends (Dunford-Schwartz come back)

Proposition. *S_1 is complete with respect to the trace norm.*

Proof. Suppose $||A_n - A_m||_1 \to 0$. Then $\exists A \in \mathcal{B}(H)$ such that $||A - A_n|| \to 0$. Suppose $k \in \mathbf{R}$, $k \geq ||A_n||_1$, $\forall n$. Let $B \in \mathcal{B}(H)$, $||B|| \leq 1$, let $\{\chi_i\}_{i \in I}$ be an orthonormal basis and let J be a finite subset of I. Then,

$$\big|\sum_{i \in J}(AB\chi_i, \chi_i)\big| = \lim_{n\to\infty} \big|\sum_{i \in J}(A_n B\chi_i, \chi_i)\big| \leq k.$$

Proposition. *S_2 is a $*$-ideal complete w.r.t. $||\ ||_2$.*

Proof. Suppose $A \in S_2$. Then we know $|A|^2 = A^*A \in S_1$. Let $U|A| = A$ be the polar decomposition of A. Then

$$AA^* = U|A|^2U^*.$$

Now, we know $||A||_2 =$

$$\begin{aligned}(\text{ Tr } A^*A)^{1/2} &= \big(\sum_i (A^*A\chi_i, \chi_i)\big)^{1/2} \\ &= \big(\sum_{ij} |(A\chi_i, \chi_j)|^2\big)^{1/2}.\end{aligned}$$

Hence,

$$\begin{aligned}||A + B||_2 &= \big(\sum_{ij} |((A+B)\chi_i, \chi_j)|^2\big)^{1/2} \\ &\leq \big(\sum_{ij} |(A\chi_i, \chi_j)|^2\big)^{1/2} + \big(\sum_{ij} |(B\chi_i, \chi_j)|^2\big)^{1/2} \\ &= ||A||_2 + ||B||_2.\end{aligned}$$

Proposition. *Suppose $(X, \mu) = ([a, b], dt)$, $k(x, y)$ is continuous and K is self-adjoint and positive. Then K is nuclear and*

$$\text{Tr}\, K = \int_a^b k(x, x)dt.$$

Idea of proof. Let $\phi_i(x)$ be an orthonormal set corresponding to $s_i(K)$. Then

$$k(x,y) = \sum_i s_i(K)\phi_i(x)\overline{\phi_i(y)}.$$

(One needs to know that one can choose the ϕ_i to be continuous (and not just in L_2 which is easy and Mercer's Theorem which asserts that then the right hand side uniformly converges to the left (Riesz-Nagy §98) to make this work.)

Highlights of the Course

The main object in this course was the spectrum $\sigma(T)$ of an operator T in a Banach algebra A. One basic fact is:

$$\text{the radius spectral of } T \leq ||T||$$

with equality if A is a C^* algebra. The Gelfand transform allowed us to think of elements in A when A is commutative as functions on a compact set (especially if A is a C^*-algebra). The Spectral Permanence Theorem asserted that if $B \subseteq A$ are unital Banach algebras, then,

$$\sigma_B(x) = \sigma_A(x) \cup \bigcup_i H_i$$

where the H_i are some holes of $\sigma_A(x)$ and $\sigma_B(x) = \sigma_A(x)$ if A and B are C^* algebras.

If N is a normal operator on a Hilbert space then the functional calculus tells us we can evaluate continuous functions f on $\sigma(N)$ at N and the spectal theorem is the assertion that there is a reasonable way to evaluate Borel functions too.

One corollary of the spectral theorem we used while studying compact operators is that if Z is a compact operator there exist positive real numbers $\mu_1(Z) \geq \mu_2(Z) \geq \cdots$ such that

$$\sum_i \mu_i(Z)(\ ,e_i)f_i = Z.$$

We spent some time trying to understand Grothendieck's paper on the Fredholm determinant and managed to generalize the main results to operators on a module over a Banach algebra. This and the result we proved about the stability of radical multiplicity sheds some light on how the spectrum changes as the operator varies.