GloptiPoly

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The problem

$$\begin{aligned} f^* \quad := \quad & \inf f(x) \\ & g_1(x) \geq 0 \\ & \vdots \\ & g_m(x) \geq 0 \end{aligned}$$

If the feasible set S is compact and nonempty: $\inf = \min$.

Set $g_0 := 1$ and $M := \{\sum_{i=0}^{m} \sigma_i g_i \,:\, \sigma_i \in \sum \mathbb{R}[x]^2\}$.

General assumption: $\exists N \,:\, N - \sum_{i=1}^{n} x_i^2 \in M$

Convexification I

Points $\rightarrow$ prob. measures: $f^* = \inf \left\{ \int f \, d\mu \ : \ \mu \in \mathcal{M}^1(S) \right\}$

Theorem (Putinar '93). For any map $L : \mathbb{R}[x] \rightarrow \mathbb{R}$ the following are equivalent:

1. L is linear, $L(1) = 1$ and $L(M) \subset [0, \infty)$.
2. $\exists \mu \in \mathcal{M}^1(S) \ : \ \forall p \in \mathbb{R}[x] \ \ L(p) = \int p \, d\mu$.

$\Longrightarrow f^* = \inf\{L(f) \ : \ L : \mathbb{R}[x] \rightarrow \mathbb{R}$ lin., $L(1) = 1, L(M) \subset [0, \infty)\}$.

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Example (n=2): L given by the moments

$L(x_1^i x_2^j) = \int x_1^i x_2^j \, d\mu$, $\quad i, j \geq 0$

Convexification II: Dual viewpoint

$$\begin{aligned} f^* &= \sup\,\{a \in \mathbb{R} \,:\, f - a \geq 0 \text{ on } S\} \\ &= \sup\,\{a \in \mathbb{R} \,:\, f - a > 0 \text{ on } S\} \end{aligned}$$

Putinar's Positivstellensatz ('93). If $p \in \mathbb{R}[x]$ satisfies $p > 0$ on S then $p \in M$.

$\Longrightarrow f^* = \sup\,\{a \in \mathbb{R} \,:\, f - a \in M\}$.

Relaxation of order k (for $k \geq \max\{\deg g_i, \deg f\}$)

Let $M_k := \{\sum_{i=0}^{m} \sigma_i g_i \ : \ \sigma_i \in \sum \mathbb{R}[x]^2 \ ,$ deg$(\sigma_i g_i) \leq k\}$.

$$
\begin{array}{lll}
(P_k :) & \min & L(f) \\
 & & L : \mathbb{R}[x]_k \to \mathbb{R} \text{ is linear} \\
 & & L(1) = 1 \\
 & & L(M_k) \subset [0, \infty)
\end{array}
$$

$$
\begin{array}{lll}
(D_k :) & \max & a \\
 & & f - a \in M_k
\end{array}
$$

Theorem (Lasserre 2001). $(D_k^*)_{k\in\mathbb{N}}$ and $(P_k^*)_{k\in\mathbb{N}}$ are increasing sequences that converge to f^* and satisfy $D_k^* \leq P_k^* \leq f^*$ for all admissible k.

Formulation as semidefinite program

Lemma. Let $L : \mathbb{R}[x]_k \to \mathbb{R}$ be a linear map. Then $L(M_k) \subset [0, \infty)$ if and only if the $m+1$ matrices

$$(L(x^{\beta+\gamma} g_i))_{(\beta,\gamma)\in s(d_i)\times s(d_i)} \qquad (0 \le i \le m)$$

are positive semidefinite, where
$d_i := \max\{d \in \mathbb{N}_0 \ : \ 2d + \deg g_i \le k\}$ and
$s(d_i) := \{\alpha \in \mathbb{N}_0^n \ : \ |\alpha| \le d_i\}$.

Example (for $g_0(x_1, x_2) = 1$, moment matrices, $y_{ij} := L(x_1^i x_2^j)$):

$$k = 2 : \left(\begin{array}{c|cc} 1 & y_{10} & y_{01} \\ \hline y_{10} & y_{20} & y_{11} \\ y_{01} & y_{11} & y_{02} \end{array}\right) \qquad k = 4 : \left(\begin{array}{c|cc|ccc} 1 & y_{10} & y_{01} & y_{20} & y_{11} & y_{02} \\ \hline y_{10} & y_{20} & y_{11} & y_{30} & y_{21} & y_{12} \\ y_{01} & y_{11} & y_{02} & y_{21} & y_{12} & y_{03} \\ \hline y_{20} & y_{30} & y_{21} & y_{40} & y_{31} & y_{22} \\ y_{11} & y_{21} & y_{12} & y_{31} & y_{22} & y_{13} \\ y_{02} & y_{12} & y_{03} & y_{22} & y_{13} & y_{04} \end{array}\right)$$

Example of second-order relaxation

$$\begin{aligned} \min \quad & 2x_1^2 - 3x_1x_2 \\ & -4x_1^2 + 7x_1 \geq 0 \\ & -x_1^2 - x_2^2 + 1 \geq 0 \end{aligned}$$

$$\rightsquigarrow \qquad \begin{aligned} \min \quad & 2y_{20} - 3y_{11} \\ & \left(\begin{array}{c|cc} 1 & y_{10} & y_{01} \\ \hline y_{10} & y_{20} & y_{11} \\ y_{01} & y_{11} & y_{02} \end{array}\right) \succeq 0 \\ & -4y_{20} + 7y_{10} \geq 0 \\ & -y_{20} - y_{02} + y_{00} \geq 0 \end{aligned}$$

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Fourth-order relaxation of first constraint:

$$\left(\begin{array}{c|cc} -4y_{20} + 7y_{10} & -4y_{30} + 7y_{20} & -4y_{21} + 7y_{11} \\ \hline -4y_{30} + 7y_{20} & -4y_{40} + 7y_{30} & -4y_{31} + 7y_{21} \\ -4y_{21} + 7y_{11} & -4y_{31} + 7y_{21} & -4y_{22} + 7y_{12} \end{array}\right) \succeq 0$$

Remarks

- Solutions of the relaxations are proven lower bounds of the minimization problem.
- Sufficient criterion for global optimality:

$$\text{rank } M_k(y^*) = \text{rank } M_{k-d}(y^*)\,.$$

  If the *equality* constraints define a 0-dim., radical ideal, this will hold for some k.
- In case rank condition holds, extraction of global optima possible by linear-algebra-techniques on polynomial equations.

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- by Didier Henrion and Jean B. Lasserre
- k’ := k/2